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Finite time blowup of multidimensional inhomogeneous isotropic Landau–Lifshitz equation on a hyperbolic space

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ABSTRACT

Blowup solutions for the multidimensional isotropic inhomogeneous Landau-Lifshitz (ILL) equation on a hyperbolic n-space $\mathcal{H}^2$ are obtained. If the inhomogeneous terms are carefully selected, the ILL will guarantee three types of singularity solutions. Type I and type II blowup solutions of any dimensional ILL on $\mathcal{H}^2$ can develop a finite time singularity from smooth initial data with finite L^2 energy. We prove the decay rate of energy density of the type III implicit blowup solution. If ε and C are non-zero constants, this solution develops a singularity in finite time with a local ($r \in [\varepsilon, C]$) finite initial energy.

1. Introduction

Spin dynamics in magnetic nanostructures is a topic of great current interest. Long wavelength spin motions in diverse ferromagnetic structures are commonly described by the phenomenological Landau-Lifschitz-Gilbert (LLG) equation. It describes the system in macroscopic language, in terms of the magnetization per unit volume $s = (s_1, s_2, s_3)$. A simple subcase of LLG is the isotropic Landau–Lifshitz–Gilbert (ILLG) equations, generalized Heisenberg models for a continuous ferromagnetic spin vector $s \in S^2 \hookrightarrow R^3$ (see, for example, [1,2])

$$s_t = \alpha s \times (\triangle s) - \beta s \times (s \times (\triangle s)). \tag{1}$$

The ILLG exhibits a rich variety of dynamical properties of a spin vector in different backgrounds. (1) is a particular case of the flow from one Riemannian manifold into another with a complex structure. The formal equivalence to a nonlinear Ginzberg–Landau equation can be seen by applying the stereographic projection from S^2 to C_∞, on the extended complex plane

$$u = \frac{s_1 + is_2}{1 + s_3}, \quad iu_t = -(\alpha - \beta i)\Delta u + \frac{2\bar{u}}{1 + |u|^2} \sum_{j=1}^{n} (\partial_j u)^2. \tag{2}$$

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In fact, the operation $\times$, which denotes the cross product in Euclidean 3-space (R^3) can be replaced by $\dot{\times}$ ($\vec{a}\dot{\times}\vec{b} = (\vec{a}\times\vec{b})\mathrm{diag}\{1,1,-1\}$) in (1). In the same time, the spin vector in R^3 is generalized to the vector in Minkowski 3-space (R^{2+1}). If the spin vector lies on the hyperbolic space $\mathcal{H}^2 = \{(s_1, s_2, s_3)\} \in R^{2+1}$, then the constraint $|s|^2 = s_1^2 + s_2^2 + s_3^2 = 1$ must be replaced by the condition $|s|^2 = s_3^2 - s_1^2 - s_2^2 = 1, s_3 > 0$, accordingly. From this point of view, (1) may be regarded as the dual ILLG [3,4],

$$s_t = \alpha s\dot{\times}(\triangle s) - \beta s\dot{\times}(s\dot{\times}(\triangle s)). \tag{3}$$

Applying the stereographic projection from $\mathcal{H}^2$ to C_∞, the hyperbolic form of (2) is

$$u = \frac{s_1 + is_2}{1 - s_3}, \quad iu_t = -(\alpha - \beta i)\Delta u - \frac{2\bar{u}}{1-|u|^2}\sum_{j=1}^{n}(\partial_j u)^2. \tag{4}$$

If the Gilbert term vanishes, (1)–(4) may be regarded as the Schrödinger map equation [5] (SME) (See [6,7] for more details about SME). The global smooth solutions of SME (or some extensional version [8] of it) are usually studied near small initial data. In [6,7], the authors were concerned with the issue of the global well-posedness of the initial-value problem SME for the case of initial data that was small in the critical Sobolev spaces. However, if the inhomogeneous property is considered, SME can be generalized into the inhomogeneous SME (ILL), first proposed by Balakrishnan (see [9]) in condensed matter physics. In this paper, we study the ILL on $\mathcal{H}^2$,

$$\frac{\partial}{\partial t}s = q(t, \overrightarrow{x})s\dot{\times}\triangle s + \nabla q(t, \overrightarrow{x})(s\dot{\times}\nabla s), \tag{5}$$

where the scalar function $q(t, \overrightarrow{x})$ is the inhomogeneous term under radially symmetrical coordinates. In this case, (5) takes the form

$$\frac{\partial}{\partial t}s - q(t,r)s\dot{\times}\left(\frac{\partial^2}{\partial r^2}s + \frac{n-1}{r}\frac{\partial}{\partial r}s\right) - \frac{\partial}{\partial r}q(t,r)\left(s\dot{\times}\frac{\partial}{\partial r}s\right) = 0, \tag{6}$$

where $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$, and $s = (s_1(t,r), s_2(t,r), s_3(t,r))$.

Do solutions of ILL (or SME) exist when there is not blowup? If the target is S^2 and the inhomogeneous term is constant, the existence results for weak solutions of ILL is established. More clearly, the local theory for classical data was established in [10,11]. Under the small data, the local and global in time of the SME with small data has been intensely studied (see [12,7,6,13,14]). Especially, the classical solution with small energy is global in time (see [7]). Comparing to the so many existence results of the S^2 case, the hyperbolic case is not so clear. Nahmod, Stefanov and Uhlenbeck prove local well posedness of the Cauchy problem for the Equivalent Form of SME on $\mathcal{H}^2$ with minimal regularity assumptions on the data and outline a method to derive well posedness of the SME from it (see [15]). Bejenaru, Ionescu, Kenig and Tataru consider equivariant solutions for the SME from 2 dimension to $\mathcal{H}^2$ with finite energy and show that they are global in time and scatter (see [16]). As further as we see, there is not any existence result (even the local solution) of the ILL on the hyperbolic space. However, these existence results (on S^2 or $\mathcal{H}^2$) of SME are expected to extend to the hyperbolic target.

We concentrate on the blowup solution of (5) (or (6)). For initial data s_0, the finite time blowup phenomenon (solution s blows up in finite time T) means that the solution s solves the problem on $[0, T] \times R^n$, where $T = T(s_0) < \infty$ is the maximal existence time of s.

The blowup dynamics of ILLG, SME, and ILL have attracted considerable attention [17]. For the SME, the possibility of a finite time blowup has been proved [5,18] near the static solution. Under polar coordinates, (1), where $\alpha = 1$ and $\beta = 0$, can be expressed as

$$s_t = s \times \left(s_{rr} + \frac{1}{r}s_r + \frac{1}{r^2}s_{\theta\theta}\right). \tag{7}$$

[5,18] prove that (7) will develop a finite time singularity near the static solution,

$$Q_1(x) = e^{\theta R}\left|\begin{matrix}\dfrac{2r}{1+r^2}\\ 0\\ \dfrac{1-r^2}{1+r^2}\end{matrix}\right. = \begin{pmatrix}\dfrac{2\cos(\theta)\,r}{1+r^2}\\ \dfrac{2\sin(\theta)\,r}{1+r^2}\\ \dfrac{1-r^2}{1+r^2}\end{pmatrix}, \quad R = \begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

Thus, solution

$$\mathcal{Q}_1 = e^{I\theta}r$$