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An error analysis of the finite element method overcoming corner singularities for the stationary Stokes problem

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ABSTRACT

In this paper, we introduce a mixed finite element method to overcome corner singularities of the stationary Stokes system on a non-convex polygon. The reference Choi and Kweon (2013) says that the velocity field and the pressure function are composed of singular parts and regular remainders near a non-convex vertex, and furthermore the regular parts are sufficiently smooth on the polygon. We use the corner singularity expansion and propose new extraction formulae for coefficients of singularities. For the proposed numerical method, we first try to find finite element solutions for the regular parts and then calculate approximations of the coefficients by the derived formulae expressed by the remainders and given functions. We show error estimates of the approximations for the regular parts and coefficients, and give some numerical experiments to confirm the efficiency and reliability of the proposed method.

1. Introduction

We are interested in a novel finite element method overcoming corner singularities of the stationary Stokes problem (cf. [1]). Solutions of boundary value problems on a (non-convex) polygon $\Omega \subset \mathbb{R}^2$ have singular behavior near corners, and such corner singularities affect the accuracy of the finite element method throughout the whole domain. Regarding the stationary Stokes problem, it is seen that singular solutions for the velocity field and pressure are composed of singular part and regular part near the corners, and the singular part is expressed by the linear combination of given leading corner singularities for the Stokes operator with no-slip boundary condition (see [2]).

In this paper, we derive new extraction formulae for coefficients of the corner singularities, called the stress intensity factors, in terms of the regular part and given external forces. They enable us to deduce a finite element method for the regular part $[\mathbf{w}, \sigma]$, so approximations of the stress intensity factors c_i and the singular solution $[\mathbf{u}, p]$ are obtained. We achieve the optimal error bound $O(h)$ for $\mathbf{w}$ in H^1. Also, we establish the error bound $O(h^{1+\lambda_1-\epsilon})$ for $\mathbf{w}$ in L^2, which implies the same error bound for c_i in the absolute value, where $\lambda_1 \in (1/2, 1)$ denotes the leading singular exponent determined by the internal angle of non-convex corner. Although this analyzed error bound for $\mathbf{w}$ in L^2 is not optimal, our numerical experiments show the optimal convergence rate 2 for the L^2-error of $\mathbf{w}$. This result is deduced from the H^2-regularity of $\mathbf{w}$.

Regarding the stationary Stokes or Navier–Stokes problem, several numerical approaches have been investigated (see [3–10]). In [11,12] finite element methods were studied with the knowledge of corner singularities, and in [13–15] numerical techniques refining triangulations were used in order to overcome the singularity of solutions. Also, some numerical strategies for singular solution were proposed by defining a formula of the stress intensity factor c_i. In [16], a global formula of c_i was introduced and a finite element solution was discussed. Then the error estimate $|c_i - c_{i,h}| = O(h)$ was obtained,

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where $c_{i,h}$ denotes the approximation of c_i. In [1], an expression of the coefficient c_i is independent of the solution $[\mathbf{u}, p]$, so the error estimate $|c_i - c_{i,h}| = O(h^{2\lambda_1-\epsilon})$ was derived for $\epsilon > 0$.

Similarly, there are various numerical strategies for the corner singularity of the second order elliptic boundary value problem (see [17–21]). Especially, the Poisson problem has been studied by several finite element methods based on a singular function expansion (see [22–28]), where the numerical methods are aimed at finding the approximations of the stress intensity factor c and regular part w. In [23,28], a singular function method was considered by adding singular function as finite element basis. In [22], a dual singular function method was introduced by giving the formula of c. In [24], finite element multigrid methods for approximating c and w were studied, and in [25] crack singularities were allowed. In [26,27], a new finite element method was proposed by giving a different formula of c in terms of only w and given functions.

Throughout this paper, we will use the following spaces and norms. For $s \geq 0$, denote by $\mathrm{H}^s(\Omega)$ the fractional order Sobolev space with the norm $\|\cdot\|_s$ (see [29–31]). We note that $\mathrm{L}^2(\Omega) = \mathrm{H}^0(\Omega)$ with the norm $\|v\|_0 = (\int_\Omega |v|^2 d\mathbf{x})^{1/2}$. We define $C_0^\infty(\Omega)$ to be the linear space of infinitely differentiable functions with compact support on Ω. For $s \geq 0$, denote by $\mathrm{H}_0^s(\Omega)$ the closure of $C_0^\infty(\Omega)$ for the norm $\|\cdot\|_s$. We have $\mathrm{H}_0^1(\Omega) = \{v \in \mathrm{H}^1(\Omega) : v|_{\partial\Omega} = 0\}$. Let $\mathrm{H}^{-s}(\Omega)$ be the dual space of $\mathrm{H}_0^s(\Omega)$ with the norm:

$$\|f\|_{-s} = \sup_{0 \neq v \in \mathrm{H}_0^s(\Omega)} \frac{\langle f, v\rangle}{\|v\|_s},$$

where $\langle , \rangle$ is the duality pairing. Let $\mathrm{L}_0^2(\Omega) = \{v \in \mathrm{L}^2(\Omega) : \int_\Omega v d\mathbf{x} = 0\}$ and $\bar{\mathrm{H}}^s(\Omega) = \mathrm{H}^s(\Omega) \cap \mathrm{L}_0^2(\Omega)$. As above, denote by $\bar{\mathrm{H}}^{-1}(\Omega)$ the dual space of $\bar{\mathrm{H}}^1(\Omega)$ with the norm $\|f\|_{\bar{\mathrm{H}}^{-1}} = \sup_{0 \neq v \in \bar{\mathrm{H}}^1(\Omega)} \langle f, v\rangle / \|v\|_1$. Also, $\mathcal{X}'$ denotes the dual space of a Banach space $\mathcal{X}$. In this paper, $C > 0$ denotes a generic positive constant only depending on Ω.

This paper is organized as follows. In Section 2, we briefly discuss singular functions for the Stokes operator with no-slip boundary condition and singular function expansion for singular solution of the Stokes equations. In Section 3, we derive new extraction formulae for the stress intensity factors c_i and give a well-posed problem for the regular part $[\mathbf{w}, \sigma]$. In Section 4, we discretize the variational form for $[\mathbf{w}, \sigma]$ and analyze error estimates for approximations of $[\mathbf{w}, \sigma]$ and c_i. In Section 5, we describe an algorithm solving the discrete problem, based on the Sherman–Morrison–Woodbury formula, and give some numerical examples to confirm analyzed convergence rates.

2. Preliminaries

In this section, we discuss a singular function expansion and regularity results regarding the stationary Stokes problem on a two-dimensional domain with corners. Let $\Omega \subset \mathbb{R}^2$ be a polygon with the boundary $\Gamma := \partial\Omega$. Consider the stationary incompressible Stokes equations as follows:

$$\begin{aligned} -\mu\Delta\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega, \\ \mathrm{div}\mathbf{u} &= g && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } \Gamma, \end{aligned} \tag{2.1}$$

where $\mathbf{u}$ is the velocity vector, p the pressure function; $\mathbf{f}$ and g are given external functions satisfying $\int_\Omega g d\mathbf{x} = 0$; $\mu > 0$ is a viscous number. In fact, the incompressible flow implies that the function g is originally zero, but in this paper we assume that g may not be zero for general purpose. This assumption provides general formulae of stress intensity factors and the general form (2.1) may be applicable to the compressible Stokes or Navier–Stokes equations (see [32–35]).

Regarding the stationary Stokes problem (2.1) on a polygon, regularity issues have been investigated. Basically, if $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $g \in \bar{\mathrm{H}}^1(\Omega)$, then the existence and uniqueness of the solution $[\mathbf{u}, p]$ in $\mathbf{H}_0^1(\Omega) \times \mathrm{L}_0^2(\Omega)$ are guaranteed (cf. Theorem 5.1 in Chapter I of [30]). Furthermore, we expect that the solution $[\mathbf{u}, p]$ belongs to $\mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$, but it does not hold for a non-convex polygon (see [2,36,37]). This situation is caused by corner singularities and such lack of regularity reduces the accuracy of finite element solution. In order to restore the accuracy, several approaches were investigated in various literatures. In particular, local mesh refinement methods have been widely used (see, for example, [38,39,15]).

Now, we introduce singular functions for the Stokes operator with no-slip boundary condition. Hereafter, it is assumed that the boundary Γ has only one non-convex vertex P placed at the origin, for simplicity (for example, see Fig. 1). Let $\omega > \pi$ be an opening angle at P, defined by $\omega := \omega_2 - \omega_1$, where ω_i are numbers satisfying $\omega_1 < \omega_2 < \omega_1 + 2\pi$. The singular functions are given by considering a non-trivial solution of the Stokes problem on a sector $\mathcal{S} := \{(r, \theta) : r > 0$ and $\theta \in (\omega_1, \omega_2)\}$ (see Section 5.1 in [40]). First, the eigenvalues λ_i related to the singular functions are roots of the trigonometric equation: $\sin^2(\lambda_i\omega) - \lambda_i^2 \sin^2\omega = 0$, where the first three eigenvalues are real, ordered by

$$\begin{aligned} &\textbf{Case 1}.\ 1/2 < \lambda_1 < \pi/\omega < \lambda_2 = 1 < \lambda_3 < 2\pi/\omega && \text{if } \omega \in (\pi, \omega_*], \\ &\textbf{Case 2}.\ 1/2 < \lambda_1 < \pi/\omega < \lambda_2 < \lambda_3 = 1 < 2\pi/\omega && \text{if } \omega \in (\omega_*, 2\pi), \end{aligned} \tag{2.2}$$

where $\omega_* \approx 1.4303\pi$ is a number satisfying $\tan\omega_* = \omega_*$. Then the singular functions $[\mathbf{\Phi}_i, \phi_i]$ corresponding to λ_i are defined by

$$\mathbf{\Phi}_i = \mu^{-1}\chi_1 r^{\lambda_i}\boldsymbol{\mathcal{T}}_i(\theta), \qquad \phi_i = \chi_1 r^{\lambda_i - 1}\xi_i(\theta), \tag{2.3}$$