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Identification of boundary conditions by solving Cauchy problem in linear elasticity with material uncertainties

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ABSTRACT

This work is a contribution to non-destructive testing in the context of uncertainties. It consists in identifying boundary conditions on an inaccessible part of a solid body boundary, from the knowledge of over-specified data given on an accessible part of this boundary. This problem is well known as Cauchy problem. The material properties are considered uncertain. Polynomial chaos expansion is used in order to solve this Cauchy problem in the random context. A stochastic constrained optimization problem is formulated and solved. Numerical experiments are presented.

1. Introduction

In this paper we are interested in the non-destructive inspection of structures in order to control the behavior and evolution of hidden boundary conditions. For example, the case of friction contact between two solids, stratified temperature in pipes, interface cracks in laminated structures etc. The measurements are often overspecified and available only on accessible parts of the structures' boundaries. For instance, measurements of full-surface displacement field (Dirichlet boundary condition) are carried out by Digital Imaging Correlation on a surface where its dual quantity field is known (Neumann boundary condition). By exploiting these data one can expand the field from the boundary inside the solid. This problem is also called data completion, it allows to use only the available data on one part of the boundary and to reconstruct the missing data on inaccessible boundaries (e.g. internal boundaries). To solve this inverse problem, the knowledge of the geometrical data, material properties are supposed to be perfect. However, in the industrial context these data are not reliable. In order to take into account their uncertain character, it is necessary to randomize them and build a stochastic model.

The data completion problem, also known as the Cauchy problem, is an ill-posed problem where the lacking data do not depend continuously on the measured data. Various ways for solving such problems can be found in the literature, in particular Tikhonov and Arsenin [1], Bui [2], Kozlov et al. [3], Marin and Lesnic [4], and Klibanov [5]. Here, we choose to use a new computational algorithm for the reconstruction of lacking data, which is based on the minimization of energy

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functional: the Constitutive Relation Error (CRE) by using overspecified data on accessible boundaries. This approach was used for thermostatic and linear elasticity problems [6–10].

The CRE is well known by its efficiency for classical identification problems [11] and was extended in the stochastic domain for modeling uncertain properties in industrial structures [12–14]. The stochastic method used was the Polynomial Chaos Expansion (PCE) [15] that was successfully applied in various domains. Among the stochastic methods dedicated to randomize the parameters of the model such as the perturbation method based on Taylor series [16] or Neumann series expansion[17,18], but unfortunately limited to problems with small random fluctuations, or large time consuming Monte Carlo method [19], PCE manages to combine efficiency for large random fluctuations problems as well as a gain time calculation. Especially, in the case of inverse problems, solving the deterministic problem is originally time consuming, and Monte Carlo simulations become oversized. PCE [15,20] is based on the representation of random processes and variables on a basis of orthogonal polynomials.

Here we propose to extend the works on the Cauchy problems [6–10] in the stochastic domain by using the PCE. The application considered here is a two-dimensional static ill-posed problem consisting in one structure with over-specified data measured on one accessible boundary and an inner boundary where the data are missing. The material properties are considered as uncertain and then modeled by random quantities. The stochastic problem is expanded on the chaos basis. Its minimization gives the random missing data (Neumann and Dirichlet). The functional is defined as the average energy error between the random solutions of two well-posed problems. The first has measured Dirichlet boundary data on the accessible part of the boundary and the unknown Neumann boundary data on the other part. The second has measured Neumann boundary data on the accessible part of the boundary and unknown Dirichlet boundary data on the other part. The minimization of the average error gives PCE unknown coefficients that allows the random solutions to be built and presented through their mean and their variance. Besides, to accelerate the convergence of such large-scale problems, the gradients of the energy functional are calculated using the stochastic adjoint problems. The method is both very efficient with respect to the accuracy of the solution and regarding the amount of computation needed. The formulation is very general and can be used with heterogeneous materials and other physical phenomena. The final solutions are given by their mean and their variance that are calculated a posteriori.

2. The deterministic data completion problem

In this section we describe briefly the data completion method for extending surface fields into an elastic solid made of a homogeneous isotropic material. This problem is known as a Cauchy problem for the Lamé operator. Let us consider the methods based on the minimization of an energy error functional as detailed in [8,9,6,7,10]. Let D be a bounded domain of an elastic solid in $\mathcal{R}^n$, $n = 2$ or 3 with piecewise Lipschitz boundary $\partial D = \Gamma_m \cup \Gamma_u$. Without loss of generality we consider $n = 2$. Γ_m and Γ_u are two open disjoint parts of ∂D, as shown on Fig. 1. On the boundary Γ_m both the Dirichlet and Neumann conditions are considered known or can be measured, whereas on the boundary Γ_u none of these conditions are known. Considering small strains assumption, the continuous problem can be written as follows:

$$\begin{cases} \nabla.\boldsymbol{\sigma}(\boldsymbol{u}) &= 0 & \text{in } D \\ \boldsymbol{\sigma}(\boldsymbol{u}) &= \boldsymbol{C} : \boldsymbol{\epsilon}(\boldsymbol{u}) & \text{in } D \\ \boldsymbol{\epsilon}(\boldsymbol{u}) &= \dfrac{1}{2}\left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^t\right) & \text{in } D \\ \boldsymbol{u} &= \boldsymbol{u}_m & \text{on } \Gamma_m \\ \boldsymbol{\sigma}(\boldsymbol{u}).n &= \boldsymbol{p}_m & \text{on } \Gamma_m \end{cases} \quad \text{with } \boldsymbol{u} \in H^1(D)^2 \text{ and } (\boldsymbol{u}_m, \boldsymbol{p}_m) \in H^{1/2}(\Gamma_m)^2 \times H^{-1/2}(\Gamma_m)^2 \tag{1}$$

where $\boldsymbol{p}_m$ is the measured surface traction vector, $\boldsymbol{u}_m$ is the measured displacement field, $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\boldsymbol{\epsilon}$ is the strain tensor and $\boldsymbol{C}$ is a constant Hooke's tensor. Solving the above problem consists in finding the missing pair data $(\boldsymbol{d}, \boldsymbol{f})$ on Γ_u such that there exists a displacement field $\boldsymbol{u}$ satisfying the following problem :

$$\begin{cases} \nabla.\boldsymbol{\sigma}(\boldsymbol{u}) &= 0 & \text{in } D \\ \boldsymbol{\sigma}(\boldsymbol{u}) &= \boldsymbol{C} : \boldsymbol{\epsilon}(\boldsymbol{u}) & \text{in } D \\ \boldsymbol{\epsilon}(\boldsymbol{u}) &= \dfrac{1}{2}\left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^t\right) & \text{in } D \\ \boldsymbol{u} &= \boldsymbol{u}_m & \text{on } \Gamma_m \\ \boldsymbol{\sigma}(\boldsymbol{u}).n &= \boldsymbol{p}_m & \text{on } \Gamma_m \\ \boldsymbol{u} &= \boldsymbol{d} & \text{on } \Gamma_u \\ \boldsymbol{\sigma}(\boldsymbol{u}).n &= \boldsymbol{f} & \text{on } \Gamma_u \end{cases} \tag{2}$$

with $\boldsymbol{u} \in H^1(D)^2$ and $(\boldsymbol{u}_m, \boldsymbol{p}_m, \boldsymbol{d}, \boldsymbol{f}) \in H^{1/2}(\Gamma_m)^2 \times H^{-1/2}(\Gamma_m)^2 \times H^{1/2}(\Gamma_u)^2 \times H^{-1/2}(\Gamma_u)^2$.

The above inverse problem is known to be generally ill-posed, i.e. the existence, uniqueness and stability of its solution are not always guaranteed, see Hadamard [21]. The uniqueness theorem for two-dimensional elasticity can be found in Muskhelishvili [22]. However, a more general version of this theorem was established by Knops et al. [23]. This problem has been studied by Yeih et al. [24,25], who have analyzed its existence, uniqueness and continuous dependence on the data (stability). To summarize, as long as the Cauchy data $\boldsymbol{u}_m$ and $\boldsymbol{p}_m$ are compatible, a unique solution of (1) exists. However,