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Two-grid method for miscible displacement problem by mixed finite element methods and finite element method of characteristics

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ABSTRACT

The miscible displacement of one incompressible fluid by another in a porous medium is governed by a system of two equations. One is elliptic form equation for the pressure and the other is parabolic form equation for the concentration of one of the fluids. Since only the velocity and not the pressure appears explicitly in the concentration equation, we use a mixed finite element method for the approximation of the pressure equation and finite element method with characteristics for the concentration equation. To linearize this full discrete scheme problems, we use two Newton iterations on the fine grid in our methods, with the initial guess coming from the coarse-grid solution. Then, we get the error estimates for the three-step two-grid algorithms. It is showed that coarse space can be extremely coarse and we achieve asymptotically optimal approximation as long as the mesh sizes satisfy $H = O(h^{\frac{1}{4}})$ in the three-step algorithm. Finally, numerical experiment indicates that two-grid algorithm is very effective.

1. Introduction

We consider the miscible displacement of one incompressible fluid by another in a reservoir $\Omega \subset \mathbb{R}^2$ of unit thickness. The nonlinear coupled system of equations that describes the pressure $p(x, t)$ and the concentration $c(x, t)$ of one of the fluids is given by

$$\phi \frac{\partial c}{\partial t} + \boldsymbol{u} \cdot \nabla c - \nabla \cdot (D \nabla c) = f(c), \tag{1.1}$$

$$\nabla \cdot \boldsymbol{u} = q, \tag{1.2}$$

$$\boldsymbol{u} = -a(c) \nabla p, \tag{1.3}$$

where $x \in \Omega$, $t \in J = [0, T]$ and $a(c) = a(x, c) = \frac{k(x)}{\mu(c)}$. $k(x)$ is the permeability of the porous rock, $\mu(c)$ is the viscosity of the fluid mixture, $\boldsymbol{u}(x, t)$ is the Darcy velocity of the mixture, $q(x, t)$ represents the flow rate at wells and q^+ is the positive

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part of the $q.f(c)$ may be nonlinear function [1]. $c_0(x)$ is the initial concentration, $\phi(x)$ is the porosity of the rock, and $D(\boldsymbol{u})$ is the coefficient of molecular diffusion and mechanical dispersion of one fluid into the other and it is the 2×2 matrix,

$$D = \phi[d_m I + |\boldsymbol{u}|(d_l E(\boldsymbol{u}) + d_t E^{\perp}(\boldsymbol{u}))],$$

where $E(\boldsymbol{u}) = \frac{\boldsymbol{u}_i \boldsymbol{u}_j}{|\boldsymbol{u}|^2}$ and $E^{\perp} = I - E$, d_m is the molecular diffusion, and d_l, d_t are, respectively, the longitudinal and transverse dispersion coefficients. For convenience, we assume that $D = \phi d_m I$ implies only the molecular diffusion and not the dispersion in this paper.

The system is subjected to boundary conditions:

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad x \in \partial\Omega, \quad t \in J, \tag{1.4}$$

$$D\nabla c \cdot \boldsymbol{n} = 0, \quad x \in \partial\Omega, \quad t \in J, \tag{1.5}$$

and an initial condition

$$c(x, 0) = c_0(x), \quad x \in \Omega. \tag{1.6}$$

Two-phase flow and transportation of fluids in porous media play an important role in both theoretic and applicative aspects. Petroleum engineers have been interested in efficient oil exploitation and the hydrologists have been concerned about improving the utilization of groundwater resource for a long time. Chen and Ewing [2] have studied the mathematical theory related to these models. In the past thirty years, numerical approximations have drawn much attention for the miscible displacement of one incompressible fluid by another in a reservoir [3–10]. Douglas and his coauthors proposed a finite element method based on the use of an elliptic mixed finite element method to approximate the pressure p and the velocity $\boldsymbol{u}$ and a parabolic Galerkin method to approximate the concentration [5]. It is not surprising that a mixed method [11,12], which computes p and $\boldsymbol{u}$ simultaneously without differentiation of p and multiplication by the rough coefficient $a(c)$, improves the approximation of c, since only the velocity and not the pressure appears explicitly in Eq. (1.1) for the concentration. Ewing and Wheeler [6] use Galerkin methods for both continuous time and full discrete approximation.

In realistic displacements D is quite small for the concentration equation, so that (1.1) for c is strongly convection-dominated. Standard Galerkin methods produce unacceptable nonphysical and oscillations in the concentration approximations. Thus, Douglas and his coauthors introduced and analyzed characteristic method for a single convection dominated diffusion equation [13]. These numerical techniques have been introduced to obtain better approximations for (1.1), such as characteristic finite difference method [10], characteristic finite element method [9], the modified method of characteristic finite element method [3,8,14,9]. In this paper, we shall approximate c by finite method of characteristics for the concentration equation.

In this paper, we try to consider an effective algorithm for this essential system. As we known, Xu proposed two-grid algorithm for nonlinear or nonsymmetric and coupled system [15–17]. It is a simple but effective algorithm that has been applied to many different kinds of problems, such as Dawson[18], Chen [19,20] make a nice work of two-grid method for quasilinear reaction diffusion equations. So, it is a natural idea to use two-grid scheme here for our model problem (1.1)–(1.3). For simplicity, we shall consider the situation where D only related to the molecular diffusion is linear and $f(c)$ is nonlinear function in our paper. We use the finite element method of characteristics for the concentration equation and mixed finite element scheme for the pressure–velocity equations. We first estimate the mixed finite element and the finite element method of characteristics solution in the sense of $L^{\infty}(J; L^{\infty})$ norm. Then, we present our main algorithm—the two-grid methods. There are lots of literatures that concern about the miscible displacement problem by different treatments, but, to the best of our knowledge, there are few results about two-grid algorithm that used to cope with such a nonlinear coupled problem. The main idea of our algorithm is to solve a nonlinear coupled system in the very coarse grid and then solve the decoupled linear system on the fine grid. It is shown that coarse grid can be coarser and still achieves asymptotically optimal approximation as long as the mesh sizes satisfy $H = O(h^{\frac{1}{4}})$ in the three-step algorithm. Under certain assumptions, we can achieve the same accuracy as the finite element method but with much less cost time since we just have to solve a small scale nonlinear problem.

The paper is organized as follows. In Section 2, we introduce a characteristic method for the concentration. In Section 3, we present the weak formulas of our model. Section 4 will make a simple analysis of the finite element solutions of the model. Our main algorithm and its convergence analysis will be advocated in Section 5. In Section 6, the numerical experiment of two-grid algorithm is presented in this paper.

2. A characteristic method for the concentration

For the concentration equation (1.1), convection essentially dominates diffusion, and it is natural to seek numerical methods for such problems that reflect their almost hyperbolic nature. We shall consider combining the method of characteristics with finite element to treat Eq. (1.1), then we shall indicate a number of extensions and applications of our concepts.

Let

$$\psi(x) = [\boldsymbol{u}^2(x) + \phi^2(x)]^{\frac{1}{2}}$$