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A numerical approach for a general class of the spatial segregation of reaction–diffusion systems arising in population dynamics

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A B S T R A C T

In the current work we consider the numerical solutions of equations of stationary states for a general class of the spatial segregation of reaction–diffusion systems with $m \geq 2$ population densities. We introduce a discrete multi-phase minimization problem related to the segregation problem, which allows to prove the existence and uniqueness of the corresponding finite difference scheme. Based on that scheme, we suggest an iterative algorithm and show its consistency and stability. For the special case $m = 2$, we show that the problem gives rise to the generalized version of the so-called two-phase obstacle problem. In this particular case we introduce the notion of viscosity solutions and prove convergence of the difference scheme to the unique viscosity solution. At the end of the paper we present computational tests, for different internal dynamics, and discuss numerical results.

1. Introduction and known results

1.1. The setting of the problem

In recent years there have been intense studies of spatial segregation for reaction–diffusion systems. The existence of spatially inhomogeneous solutions for competition models of Lotka–Volterra type in the case of two and more competing densities has been considered in [1–7]. The aim of this paper is to study the numerical solutions for a certain class of the spatial segregation of reaction–diffusion system with m population densities.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a connected and bounded domain with smooth boundary and m be a fixed integer. We consider the steady-states of m competing species coexisting in the same area Ω. Let u_i denotes the population density of the ith component with the internal dynamic prescribed by F_i.

We call the m-tuple $U = (u_1, \dots, u_m) \in (H^1(\Omega))^m$, *a segregated state* if

$$u_i(x) \cdot u_j(x) = 0, \text{ a.e.} \quad \text{for } i \neq j,\ x \in \Omega.$$

The problem amounts to

$$\text{Minimize } E(u_1, \dots, u_m) = \int_\Omega \sum_{i=1}^{m} \left(\frac{1}{2} |\nabla u_i(x)|^2 + F_i(x, u_i(x)) \right) dx \tag{1}$$

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over the set

$$S = \{(u_1, \dots, u_m) \in (H^1(\Omega))^m : u_i \geq 0, u_i \cdot u_j = 0, u_i = \phi_i \text{ on } \partial\Omega\},$$

where $\phi_i \in H^{\frac{1}{2}}(\partial\Omega)$, $\phi_i \cdot \phi_j = 0$, for $i \neq j$ and $\phi_i \geq 0$ on the boundary $\partial\Omega$.

We assume that

$$F_i(x, s) = \int_0^s f_i(x, v)dv,$$

where $f_i(x, s) : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ is Lipschitz continuous in s, uniformly continuous in x and $f_i(x, 0) \equiv 0$.

Remark 1. Functions f_i's are defined only for non negative values of s (recall that our densities u_i's are assumed non negative); thus we can arbitrarily define such functions on the negative semiaxis. For the sake of convenience, when $s \leq 0$, we will let $f_i(x, s) = -f_i(x, -s)$. This extension preserves the continuity due to the conditions on f_i defined above. In the same way, each F_i is extended as an even function.

Remark 2. We emphasize that for the case $f_i(x, s) = f_i(x)$, the assumption is that for all i the functions f_i are nonnegative and uniformly continuous in x. Also for simplicity, throughout the paper we shall call both F_i and f_i as internal dynamics.

Remark 3. We would like to point out that the only difference between our minimization problem (1) and the problem discussed by Conti, Terracini and Verzini [2], is the sign in front of the internal dynamics F_i. In our case, the plus sign of F_i allows to get rid of some additional conditions, which are imposed in [2, Section 2]. Those conditions are important to provide coercivity of a minimizing functional in [2]. But in our case the above given conditions together with convexity assumption on $F_i(x, s)$, with respect to the variable s are enough to conclude $F_i(x, u_i(x)) \geq 0$, which in turn implies coercivity of a functional (1).

In order to speak on the local properties of the population densities, let us introduce the notion of multiplicity of a point in Ω.

Definition 1. The multiplicity of the point $x \in \overline{\Omega}$ is defined by:

$$m(x) = \text{card}\,\{i : measure(\Omega_i \cap B(x, r)) > 0, \forall r > 0\},$$

where $\Omega_i = \{u_i > 0\}$.

For the local properties of u_i the same results as in [2] with the opposite sign in front of the internal dynamics f_i hold. Below, for the sake of clarity, we write down those results from [2] with appropriate changes.

Lemma 1 (*Proposition 6.3 in [2]*). *Assume that $x_0 \in \Omega$, then the following holds:*

(1) *If $m(x_0) = 0$, then there exists $r > 0$ such that for every $i = 1, \dots, m$;*

$$u_i \equiv 0 \quad \text{on } B(x_0, r).$$

(2) *If $m(x_0) = 1$, then there are i and $r > 0$ such that in $B(x_0, r)$*

$$\Delta u_i = f_i(x, u_i), \quad u_j \equiv 0 \text{ for } j \neq i.$$

(3) *If $m(x_0) = 2$, then there are i, j and $r > 0$ such that for every k and $k \neq i, j$, we have $u_k \equiv 0$ and in $B(x_0, r)$*

$$\Delta(u_i - u_j) = f_i(x, (u_i - u_j))\chi_{\{u_i > u_j\}} - f_j(x, -(u_i - u_j))\chi_{\{u_i < u_j\}}.$$

Next, we state the following uniqueness theorem due to Conti, Terracini and Verzini.

Theorem 1 (*Theorem 4.2 in [2]*). *Let the functional in minimization problem (1) be coercive and moreover each $F_i(x, s)$ be convex in the variable s, for all $x \in \Omega$. Then, the problem (1) has a unique minimizer.*

This theorem will play a crucial role in studying the difference scheme, especially for the case $m = 2$ where we will reformulate it as a generalized two-phase obstacle problem. Note that in this case, the problem will be reduced to:

$$\text{Minimize } E(u_1, u_2) = \int_\Omega \sum_{i=1}^{2} \left(\frac{1}{2}|\nabla u_i(x)|^2 + F_i(x, u_i(x)) \right) dx, \tag{2}$$

over the set

$$S = \{(u_1, u_2) \in (H^1(\Omega))^2 : u_i \geq 0, u_1 \cdot u_2 = 0, u_i = \phi_i \text{ on } \partial\Omega\}.$$

Here $\phi_i \in H^{\frac{1}{2}}(\partial\Omega)$ with property $\phi_1 \cdot \phi_2 = 0$, $\phi_i \geq 0$ on the boundary $\partial\Omega$.

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