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On the time-fractional Navier–Stokes equations[☆]

Yong Zhou [a,b,*], Li Peng [a]

[a] Faculty of Mathematics and Computational Science, Xiangtan University, Hunan 411105, PR China

[b] Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia

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ABSTRACT

This paper is concerned with the Navier–Stokes equations with time-fractional derivative of order $\alpha \in (0, 1)$. This type of equations can be used to simulate anomalous diffusion in fractal media. We establish the existence and uniqueness of local and global mild solutions in $H^{\beta,q}$. Meanwhile, we also give local mild solutions in J_q. Moreover, we prove the existence and regularity of classical solutions for such equations in J_q.

1. Introduction

The Navier–Stokes equations describe the motion of the incompressible Newtonian fluid flows ranging from large scale atmospheric motions to the lubrication of ball bearings, and express the conservation of mass and momentum. For more details we refer to the monographs of Cannone [1] and Varnhorn [2]. We find this system which is so rich in phenomena that the whole power of mathematical theory is needed to discuss the existence, regularity and boundary conditions; see, e.g., Lemarié-Rieusset [3] and Von Wahl [4].

It is worth mentioning that Leray carried out an initial study that a boundary-value problem for the time-dependent Navier–Stokes equations possesses a unique smooth solution on some intervals of time provided the data are sufficiently smooth. Since then many results on the existence for weak, mild and strong solutions for the Navier–Stokes equations have been investigated intensively by many authors; see, e.g., Almeida and Ferreira [5], Heck et al. [6], Iwabuchi and Takada [7], Koch et al. [8], Masmoudi and Wong [9], and Weissler [10]. Moreover, one can find results on regularity of weak and strong solution from Amrouche and Rejaiba [11], Chemin and Gallagher [12], Chemin et al. [13], Choe [14], Danchin [15], Giga [16], Kozono [17], Raugel and Sell [18] and the references therein.

On the other hand, fractional calculus has gained considerable popularity during the past decades due mainly to its demonstrated applications in numerous seemingly diverse and wide-spread fields of science and engineering, including fluid flow, rheology, dynamical processes and porous structures, diffusive transport akin to diffusion, control theory of dynamical systems, viscoelasticity and so on; see, e.g., Herrmann [19], Hilfer [20] and Zhou et al. [21–23]. The most important among such models are those described by partial differential equations with fractional derivatives. Such models are interesting for not only physicists but also pure mathematicians.

Recent theoretical analysis and experimental data have shown that classical diffusion equation fails to describe diffusion phenomenon in heterogeneous porous media that exhibits fractal characteristics. How is the classical diffusion equation

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* Corresponding author at: Faculty of Mathematics and Computational Science, Xiangtan University, Hunan 411105, PR China.
E-mail address: yzhou@xtu.edu.cn (Y. Zhou).

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modified to make it appropriate to depict anomalous diffusion phenomena? This problem is interesting for researchers. Fractional calculus have been found effective in modelling anomalous diffusion processes since it has been recognized as one of the best tools to characterize the long memory processes. Consequently, it is reasonable and significative to propose the generalized Navier–Stokes equations with Caputo fractional derivative operator, which can be used to simulate anomalous diffusion in fractal media. Its evolutions behave in a much more complex way than in classical inter-order case and the corresponding investigation becomes more challenging.

The main effort on time-fractional Navier–Stokes equations has been put into attempts to derive numerical solutions and analytical solutions; see Ganji et al. [24], El-Shahed et al. [25], and Momani and Zaid [26]. However, to the best of our knowledge, there are very few results on the existence and regularity of mild solutions for time-fractional Navier–Stokes equations. Recently, Carvalho-Neto [27] dealt with the existence and uniqueness of global and local mild solutions for the time-fractional Navier–Stokes equations.

Motivated by above discussion, in this paper we study the following time-fractional Navier–Stokes equations in an open set $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$:

$$\begin{cases} \partial_t^\alpha u - \nu\Delta u + (u\cdot\nabla)u = -\nabla p + f, \quad t>0, \\ \nabla\cdot u = 0, \\ u|_{\partial\Omega} = 0, \\ u(0,x) = a, \end{cases} \tag{1.1}$$

where ∂_t^α is the Caputo fractional derivative of order $\alpha \in (0,1)$, $u = (u_1(t,x), u_2(t,x), \ldots, u_n(t,x))$ represents the velocity field at a point $x \in \Omega$ and time $t > 0$, $p = p(t,x)$ is the pressure, ν the viscosity, $f = f(t,x)$ is the external force and $a = a(x)$ is the initial velocity. From now on, we assume that Ω has a smooth boundary.

Firstly, we get rid of the pressure term by applying Helmholtz projector P to Eq. (1.1), which converts Eq. (1.1) to

$$\begin{cases} \partial_t^\alpha u - \nu P\Delta u + P(u\cdot\nabla)u = Pf, \quad t>0, \\ \nabla\cdot u = 0, \\ u|_{\partial\Omega} = 0, \\ u(0,x) = a. \end{cases}$$

The operator $-\nu P\Delta$ with Dirichlet boundary conditions is, basically, the Stokes operator A in the divergence-free function space under consideration. Then we rewrite (1.1) as the following abstract form

$$\begin{cases} {}^C D_t^\alpha u = -Au + F(u,u) + Pf, \quad t>0, \\ u(0) = a, \end{cases} \tag{1.2}$$

where $F(u, v) = -P(u\cdot\nabla)v$. If one can give sense to the Helmholtz projection P and the Stokes operator A, then the solution of Eq. (1.2) is also the solution of Eq. (1.1).

The objective of this paper is to establish the existence and uniqueness of global and local mild solutions of problem (1.2) in $H^{\beta,q}$. Further, we prove the regularity results which state essentially that if Pf is Hölder continuous then there is a unique classical solution $u(t)$ such that Au and ${}^C D_t^\alpha u(t)$ are Hölder continuous in J_q.

The paper is organized as follows. In Section 2 we recall some notations, definitions, and preliminary facts. Section 3 is devoted to the existence and uniqueness of global mild solution in $H^{\beta,q}$ of problem (1.2), then proceed to study the local mild solution in $H^{\beta,q}$. In Section 4, we use the iteration method to obtain the existence and uniqueness of local mild solution in J_q of problem (1.2). Finally, Section 5 is concerned with the existence and regularity of classical solution in J_q of problem (1.2).

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let $\Omega = \{(x_1, \ldots, x_n) : x_n > 0\}$ be open subset of $\mathbb{R}^n$, where $n \geq 3$. Let $1 < q < \infty$. Then there is a bounded projection P called the Hodge projection on $(L^q(\Omega))^n$, whose range is the closure of

$$C_\sigma^\infty(\Omega) := \{u \in (C^\infty(\Omega))^n : \nabla\cdot u = 0,\ u \text{ has compact support in } \Omega\},$$

and whose null space is the closure of

$$\{u \in (C^\infty(\Omega))^n : u = \nabla\phi,\ \phi \in C^\infty(\Omega)\}.$$

For notational convenience, let $J_q := \overline{C_\sigma^\infty(\Omega)}^{|\cdot|_q}$, which is a closed subspace of $(L^q(\Omega))^n$. $(W^{m,q}(\Omega))^n$ is a Sobolev space with the norm $|\cdot|_{m,q}$.

$A = -\nu P\Delta$ denotes the Stokes operator in J_q whose domain is $D_q(A) = D_q(\Delta) \cap J_q$; here,

$$D_q(\Delta) = \{u \in (W^{2,q}(\Omega))^n : u|_{\partial\Omega} = 0\}.$$

It is known that $-A$ is a closed linear operator and generates the bounded analytic semigroup $\{e^{-tA}\}$ on J_q.

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