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Applications of inverse tempered stable subordinators

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ABSTRACT

The inverse tempered stable subordinator is a stochastic process that models power law waiting times between particle movements, with an exponential tempering that allows all moments to exist. This paper shows that the probability density function of an inverse tempered stable subordinator solves a tempered time-fractional diffusion equation, and its "folded" density solves a tempered time-fractional telegraph equation. Two explicit formulae for the density function are developed, and applied to compute explicit solutions to tempered fractional Cauchy problems, where a tempered fractional derivative replaces the first derivative in time. Several examples are given, including tempered fractional diffusion equations on bounded or unbounded domains, and the probability distribution of a tempered fractional Poisson process. It is shown that solutions to the tempered fractional diffusion equation have a cusp at the origin.

1. Introduction

Fractional calculus is a very old field, dating back to a letter from Leibniz to L'Hôpital in 1695. In recent decades, the subject has expanded rapidly, due to the discovery of interesting mathematical connections, and real world applications. See for example [1–8]. The close connection between fractional calculus and probability is outlined in [9–11]. A famous paper of Einstein [12] outlined the classical link between random walks, Brownian motion, and the diffusion equation. In the modern theory, the probability of a jump exceeding length x falls off like a power law $x^{-\alpha}$ for some $0 < \alpha < 2$, the random walk limit is an α-stable Lévy motion whose particle traces are fractals of dimension α, and whose particle density solves a diffusion equation involving a fractional derivative of order α in the space variable. If particles wait a random time between jumps, with a probability that falls off like $t^{-\beta}$ for some $0 < \beta < 1$, the non-Markovian limit density solves a space–time fractional diffusion equation that involves a fractional derivative of order β in the time variable. Particle traces follow a random process obtained by replacing the time variable in the α-stable Lévy motion by an inverse β-stable subordinator. The fractal dimension α of the particle paths remains the same, since the inverse stable subordinator is continuous and nondecreasing [13].

The space–time fractional diffusion model implies that the mean waiting time, and the second moment of the particle jump distribution, are both infinite. The tempered fractional diffusion model was developed as an alternative with finite moments [14–19]. This model has proven useful in applications to geophysics [20–23] and finance [24,25]. Fractional Cauchy

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problems govern the long time limiting behavior of particle motions [26–28], assuming a power law distributed waiting time between movements, in a general framework that can accommodate complex boundary conditions and confining potentials. Tempered fractional Cauchy problems modify this general model, tempering the power law waiting times, so that the mean waiting time remains finite [29]. A tempered fractional derivative in time replaces the usual first derivative in the classical Cauchy problem. The Cauchy problem governs a Markov process, but the tempered fractional Cauchy problem governs a non-Markovian process, since the resting times are not exponentially distributed. Particle motions follow a time-changed Markov process, using an inverse tempered stable subordinator introduced in [30] as a subdiffusion model with finite moments. The inverse tempered subordinator was applied to a tempered fractional Fokker–Planck equation in [31] and applied to financial data [32,33].

The goal of this paper is to develop properties of the inverse tempered stable subordinator, to facilitate practical applications of tempered fractional Cauchy problems. Section 2 reviews some basic facts about tempered fractional calculus. In Section 3, we show that the probability density function of an inverse tempered stable subordinator solves a tempered time-fractional diffusion equation, and its "folded" density solves a tempered time-fractional telegraph equation. In Section 4, we develop two explicit formulae for the inverse tempered stable density. In Section 5, we prove scaling and asymptotic properties for the tempered stable subordinator and its inverse. Section 6 applies the inverse tempered stable density to solve several tempered fractional Cauchy problems, including tempered fractional diffusion equations on bounded and unbounded domains. There we also prove that solutions to the tempered fractional diffusion equation are non-differentiable at the center of mass. Section 7 discusses the tempered fractional Poisson process, and applies a formula from Section 4 to compute its probability distribution.

2. Tempered fractional calculus

The standard β-stable subordinator D_x is a Lévy process (i.e., it has stationary and independent increments) whose probability density function (pdf) $g(t,x)$ has Laplace transform

$$\tilde{g}(s,x) = \int_0^\infty e^{-st} g(t,x)\, dt = e^{-xs^\beta} \tag{2.1}$$

for some $0 < \beta < 1$. It follows easily that

$$g^\lambda(t,x) := e^{-\lambda t} g(t,x) e^{x\lambda^\beta} \tag{2.2}$$

is also a pdf. In fact, it is infinitely divisible [9, p. 208], and there is another Lévy process D_x^λ called a *tempered stable subordinator* with pdf $g^\lambda(t,x)$ for each $t > 0$. Using (2.1) it follows that

$$\tilde{g}^\lambda(s,x) = e^{-x\psi^\lambda(s)} \tag{2.3}$$

where the *Laplace symbol*

$$\psi^\lambda(s) = (s+\lambda)^\beta - \lambda^\beta. \tag{2.4}$$

Taking derivatives in (2.3) yields

$$\partial_x \tilde{g}^\lambda(s,x) = -\psi^\lambda(s)\tilde{g}^\lambda(s,x). \tag{2.5}$$

Define the *Riemann–Liouville tempered fractional derivative*

$$\mathbb{D}_t^{\beta,\lambda} g(t) = e^{-\lambda t}\, \mathbb{D}_t^\beta \left[e^{\lambda t} g(t)\right] - \lambda^\beta g(t), \tag{2.6}$$

where

$$\mathbb{D}_t^\beta g(t) = \frac{1}{\Gamma(1-\beta)} \frac{d^n}{dt^n} \int_0^t \frac{g(s)\, ds}{(t-s)^{\beta+1-n}}$$

is the usual Riemann–Liouville fractional derivative of order $\beta > 0$, and $n = \lceil \beta \rceil$ is the ceiling function, so that $n-1 < \beta \le n$. A simple argument [9, p. 209] using the shift property of the Laplace transform shows that

$$\mathcal{L}[\mathbb{D}_t^{\beta,\lambda} g](s) = \int_0^\infty e^{-st}\, \mathbb{D}_t^{\beta,\lambda} g(t)\, dt = \psi^\lambda(s)\tilde{g}(s), \tag{2.7}$$

and then inverting the Laplace transform in (2.5) shows that the tempered stable subordinator pdf solves the tempered fractional diffusion equation

$$\partial_x g^\lambda(t,x) = -\mathbb{D}_t^{\beta,\lambda} g^\lambda(t,x).$$

Note that, while the argument in [9, p. 209] assumes $0 < \beta < 1$, exactly the same argument goes through for any $\beta > 0$. Hence the definition (2.6), and the Laplace transform formula (2.7), are valid for any $\beta > 0$.

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