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On the relation between sources and initial conditions for the wave and diffusion equations

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ABSTRACT

In seismic forward modeling and fluid-flow simulations, sources and initial conditions are two approaches to initiate the perturbation of the medium in order to compute synthetic seismograms and pore-pressure maps of a reservoir, respectively. Assuming delta functions in time and space, source and initial particle velocity are equivalent in the first case (wave equation), while in the second case (diffusion equation) source and diffusion field are equivalent. The differential equation based on fractional derivatives unifies both cases but those equivalences break down when the order of the derivative is not a natural number. A simulation example, based on the fractional wave equation, illustrates the implementation of the source with a band-limited time history on a numerical mesh. In this case, the implementation of the initial condition requires a numerical calculation since the medium is heterogeneous. Body forces and stress sources (e.g., earthquakes) can easily be related in uniform media. A 1D example shows such a relation.

1. Introduction

The correct source modeling in direct grid methods is essential in hydrocarbon prospecting and earthquake seismology to compute synthetic seismograms, as well as in reservoir simulations, where the governing equations are based on the diffusion equation [1,2]. The initiation of the wavefield by sources or initial conditions is partially outlined in the literature but not fully detailed and verified. In most cases, there is not even the complete information (e.g., wave shape, frequency) to reproduce the results, yet being this aspect of numerical modeling essential.

We show in this work the equivalences and the differences between the two approaches by using the 1D differential equation, which allows us to provide simple but insightful mathematical demonstrations. First, we consider the wave equation with a source term (the inhomogeneous equation) and initial conditions. We obtain the solution by performing a double Fourier transform to the wavenumber-Laplace domain, and by means of the residue theorem we get the explicit closed-form solution in the space–time domain. The same procedure is applied to the diffusion equation. We generalize the approach by considering the fractional differential equations, which include both the wave and diffusion equation.

Details of the source implementation in a 2D full-wave modeling algorithm based on the fractional differential equation are shown, where the spatial derivatives are computed with the Fourier pseudospectral method and the fractional time

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derivative is approximated with the Grünwald–Letnikov series [3]. The Fourier method used here is accurate (negligible numerical dispersion) up to the maximum wavenumber of the mesh that corresponds to a spatial wavelength of two grid points [2]. A final 1D example shows the relation between body forces and stress sources.

2. The wave equation

Let us consider the 1D wave equation

$$u_{,tt} - c^2 u_{,xx} = S(x,t), \quad -\infty < x < +\infty,\ t \geq 0, \tag{1}$$

where $u = u(x,t)$ is the response variable (e.g., the displacement field), $c > 0$ is a characteristic velocity,

$$S(x,t) = S_0 \delta(x)\delta(t), \tag{2}$$

is the source term and δ denotes the Dirac generalized function [2].

We consider the initial conditions (suitable for a PDE of the second order in time)

$$\begin{aligned} u(x,0) &= U_0 \delta(x), \\ u_{,t}(x,0) &= V_0 \delta(x). \end{aligned} \tag{3}$$

To be clear, let us perform a dimensional analysis of the above equations. We have

$$\begin{aligned} &c = \left[LT^{-1}\right], \qquad S = \left[T^{-2}\right], \qquad \delta(x) = \left[L^{-1}\right], \qquad \delta(t) = \left[T^{-1}\right], \\ &S_0 = \left[LT^{-1}\right], \qquad U_0 = [L], \qquad V_0 = \left[LT^{-1}\right]. \end{aligned}$$

We denote the solutions separately due to S_0, U_0 and V_0 by $\mathcal{G}^w_*(x,t)$, $\mathcal{G}^w_{1C}(x,t)$ and $\mathcal{G}^w_{2C}(x,t)$, respectively. They are referred to as the *Green functions* for the corresponding Boundary Value Problems of the wave equation. The upper index w refers generically to the wave equation, the lower index "*" refers to the non-homogeneous wave equation with the source term and both vanishing initial conditions, while 1C and 2C refer to the homogeneous wave equation with the first and the second non vanishing initial condition, respectively (the so-called first and second Cauchy problems).

Let us introduce the Fourier transform with respect to x and the Laplace transform with respect to t of the response variable using the following notation

$$\widehat{u}(\kappa,t) := \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\kappa x}\, u(x,t)\,\mathrm{d}x, \qquad \widetilde{u}(x,t) := \int_0^{\infty} \mathrm{e}^{-st}\, u(x,t)\,\mathrm{d}t,$$

where κ is the wavenumber, s is the Laplace variable and $\mathrm{i} = \sqrt{-1}$.

Applying the Fourier transform to (1) gives

$$\widehat{u}_{,tt} + c^2\kappa^2 \widehat{u} = S_0 \delta(t), \tag{4}$$

Applying the Laplace transform to this equation yields

$$s^2 \widetilde{\widehat{u}} - U_0 s - V_0 + c^2\kappa^2 \widetilde{\widehat{u}} = S_0. \tag{5}$$

Eq. (5) gives the combined Laplace–Fourier transform of the solution of (1) with conditions (2) and (3):

$$\widetilde{\widehat{u}}(\kappa,s) = \frac{U_0 s + V_0 + S_0}{s^2 + c^2\kappa^2}. \tag{6}$$

From this result, we already see that the initial condition $V_0 \neq 0$ provides a Green function similar to that provided by the source $S_0 \neq 0$, that is $\mathcal{G}^w_*(x,t) \propto \mathcal{G}^w_{2C}(x,t)$.

To be clear, a check of the correctness of the dimensional analysis of Eq. (6) can be done taking into account

$$\kappa = \left[\frac{1}{L}\right], \qquad s = \left[\frac{1}{T}\right], \qquad \widetilde{\widehat{u}} = [LT].$$

It can be seen that (6) satisfies the dimensional analysis.

Let us now consider $U_0 = V_0 = 0$ in order to derive the Green function $\mathcal{G}^w_*(x,t)$ corresponding to the source. For this aim let us first perform the inverse Fourier transform of (6). We have

$$\widetilde{u}(x,s) = \frac{S_0}{2\pi c^2} \int_{-\infty}^{\infty} \frac{\exp(\mathrm{i}\kappa x)}{\kappa^2 + s^2/c^2}\,\mathrm{d}\kappa. \tag{7}$$

We use the residue theorem to solve this equation. The denominator has two poles, $\kappa = \mathrm{i}s/c$ and $\kappa = -\mathrm{i}s/c$, which correspond to the residues

$$\frac{\exp(-sx/c)}{2\mathrm{i}s/c} \quad \text{and} \quad -\frac{\exp(-sx/c)}{2\mathrm{i}s/c},$$

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