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Existence and regularity of solutions to time-fractional diffusion equations

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ABSTRACT

We investigate some initial–boundary value problems for time-fractional diffusion equations of order $\alpha \in (0, 1)$. Such equations model anomalous diffusion on fractals. The existence of solution irrelevant to α is established only if the external force function f is weighted Hölder continuous, which is weaker than Hölder continuous. Some interesting versions of maximal and spatial regularity criteria depending on the fractional exponent α are also discussed.

1. Introduction

In the present paper, as a first problem, we study the existence and regularity of solutions to the following initial value problem

$$\begin{cases} {}^cD_t^\alpha u(x,t) + \mathcal{A}u(x,t) = f(x,t) & \text{in } \mathbb{R}^n \times (0,b), \\ u(x,0) = u_0(x), & \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where ${}^cD_t^\alpha$ is the Caputo fractional derivative of order $\alpha \in (0, 1)$ with respect to t (see [1]), $f \in \mathcal{F}^{q,\sigma}((0, b], H^{-1}(\mathbb{R}^n))$, $\mathcal{F}^{q,\sigma}$ is a space of weighted Hölder continuous functions to be defined later, $0 < \sigma < q \le 1$, u_0 is a given initial data for u,

$$\mathcal{A}u(x,t) = -\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\sum_{j=1}^n a_{ij}(x)\frac{\partial}{\partial x_j}u(x,t)\right) + b(x)u(x,t), \tag{1.2}$$

the real-valued functions a_{ij} satisfy

$$a_{ij} \in L^\infty(\mathbb{R}^n), \quad 1 \le i,j \le n, \tag{1.3}$$

$$C_0\sum_{i=1}^n \xi_i^2 \le \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j, \quad \text{a.e. } x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n, \tag{1.4}$$

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with some constant $C_0 > 0$, and $b(x)$ is a real-valued function satisfying

$$b \in L^{\infty}(\mathbb{R}^n) \quad \text{and} \quad b(x) \geq B_0 > 0, \quad \text{a.e. } x \in \mathbb{R}^n. \tag{1.5}$$

The second problem deals with the existence and regularity of solutions for the Dirichlet type initial–boundary-value problem

$$\begin{cases} {}^cD_t^{\alpha} u(x,t) + \mathcal{A}u(x,t) = f(x,t) & \text{in } \Omega \times (0,b), \\ u = 0, & \text{on } \partial\Omega \times (0,b), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.6}$$

where $\Omega \subset \mathbb{R}^n, f \in \mathcal{F}^{q,\sigma}((0,b], H^{-1}(\Omega))$, $0 < \sigma < q \leq 1$, u_0 is a given initial data for u, $\mathcal{A}$ is the same as (1.2), the real-valued functions a_{ij} satisfy (1.3)–(1.4) on $\Omega \subset \mathbb{R}^n$, and real-valued function $b(x)$ satisfies (1.5).

Thirdly, we investigate the existence and regularity of solutions to an initial–boundary-value problem of Neumann-type

$$\begin{cases} {}^cD_t^{\alpha} u(x,t) + \mathcal{A}u(x,t) = f(x,t) & \text{in } \Omega \times (0,b), \\ \displaystyle\sum_{i,j=1}^{n} \nu_i(x) a_{ij}(x) \frac{\partial}{\partial x_j} u(x,t) = 0, & \text{on } \partial\Omega \times (0,b), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.7}$$

where $\Omega \subset \mathbb{R}^n, f \in \mathcal{F}^{q,\sigma}((0,b], H^1(\Omega)^*)$, $0 < \sigma < q \leq 1$, u_0 is a given initial data for u, $\mathcal{A}$ is the same as (1.2), real-valued functions a_{ij} satisfy (1.3)–(1.4) on $\Omega \subset \mathbb{R}^n$, and real-valued function $b(x)$ satisfies (1.5).

In the mathematical modeling of physical phenomena, fractional differential equations are found to be better tools than their corresponding integer-order counterparts, for example, the description of anomalous diffusion via such equations leads to more informative and interesting model [2]. It has been due to the nonlocal nature of fractional-order operators which can take into account the hereditary characteristics of the phenomena and processes involved in the modeling of real world problems. It has been observed that the mean square displacement of a diffusive material is proportional to t when $t \to \infty$ in normal diffusion (integer-order diffusion), but this proportionality becomes of the order t^{α} for $t \to \infty$ in case of anomalous diffusion. This aspect clearly indicates the importance and adaptation of fractional-order operators in the mathematical modeling of scientific and technical problems. For instance, in the study of fractional advection–dispersion equation and fractional Fokker–Planck equation [3,4], a laboratory tracer test has indicated that fractional differential equations approximate Lévy motion better than integer-order equations. For more details, see [5–12].

In [13], the author proved a maximum principle for the generalized time-fractional diffusion equation on an open bounded domain (like in (1.6)) and then applied it to show the existence of at most one classical solution for an initial–boundary-value problem involving this equation, while the uniqueness and existence results for the same equation were obtained in [14]. The stability properties and uniqueness of solutions for the homogeneous generalized time-fractional diffusion equation ((1.6) with $f = 0$) were discussed in [15]. Some recent results on the topic can be found in [16–19].

It is natural to investigate a usual initial value problem or initial–boundary value problem from the physical point of view. The initial condition $u(x,0) = u_0(x)$ determines the necessity to use the Caputo fractional derivative. In case we take Riemann–Liouville fractional derivative, the proper initial data will be the limiting value of the fractional integral of solution of order $1-\alpha$ as $t \to 0$, but it will not be the limiting value of the solution itself (see [1]). A classical solution for the initial value problem of fractional diffusion equation has been constructed and studied in [20] when the bounded function f is jointly continuous in (x,t) and locally Hölder continuous in x, and the coefficients of $\mathcal{A}$ are bounded Hölder continuous. In [21], the author obtained an interior Hölder estimate for a bounded weak solution to an initial value problem and the L^{∞} bound of the solution. In [22], a maximum principle for linear diffusion equation with multiple fractional time derivatives was established and a generalized solution with $f = 0$ was constructed. In [23], some basic properties such as existence and regularity of solution to an initial–boundary value problem for linear diffusion equation with multiple fractional time derivatives were discussed.

There are also several papers devoted to abstract form of the equations considered in our paper, see [24–26], where the existence of classical solution is obtained under Hölder condition.

Assuming f to be weighted Hölder continuous, which is weaker than Hölder continuous, we show the existence of classical solutions to the abstract form of (1.1), (1.6) and (1.7). This conclusion extends and partly improves the results in [24,25]. For more results involving weighted Hölder continuous functions, we refer the reader to the works [27–30] and the references therein. Furthermore, if f is assumed to be weighted Hölder continuous, then ${}^cD_t^{\alpha} u$ and $\mathcal{A}u$ also belong to the function space for f. In this scenario, the regularity of solutions is termed as maximal regularity. On the other hand, if f possesses spatial regularity, then the classical solutions of (1.1), (1.6) and (1.7) do have the same regularity.

This paper is organized as follows. Section 2 provides the definitions and preliminary results to be used in the sequel. In Section 3, the existence and regularity results for (1.1), (1.6) and (1.7) are investigated. We also derive some new estimations for regularity.