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Solution of stochastic nonlinear time fractional PDEs using polynomial chaos expansion combined with an exponential integrator

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ABSTRACT

The aim of this paper is to introduce an efficient numerical algorithm for the solution of stochastic time fractional stiff partial differential equations (PDEs). The time fractional derivative is described in the Caputo sense. By applying polynomial chaos (PC) expansion to discretize the random variable, a coupled time fractional deterministic system of PDEs is obtained. For the resulting system of time fractional PDEs we apply the Fourier spectral method to discretize the spatial variable, and use the fast Fourier transform (FFT) during the computation. Then, we use an exponential integrator method to overcome the stability issues due to the stiffness in the resulting time fractional semi-discrete system. The considered models have two challenging parameters, fractional order of equations and noise amplitude. In addition to exponential integrator, we also implemented a predictor–corrector method of Adams–Bashforth–Moulton type. We include numerical results of applying the developed method to stochastic time fractional Burgers and Kuramoto–Sivashinsky (KS) equations, for various values of noise intensity. A comparison of performance of the proposed scheme with fractional Adams method is also reported which confirms the efficiency and applicability of the proposed method based on the exponential integrator scheme.

1. Introduction

The numerical integration of large systems of stiff ordinary differential equations is one of the primary concerns in the field of scientific computing and numerical analysis. If the ratio of largest to smallest (in magnitude) eigenvalues of differentiation matrix applied to discretize the spatial derivatives of a PDE be very large or if a PDE has spatial derivatives of higher than second order, then the problem is more likely to be stiff [1]. In these kinds of problems, different physical phenomena may act on very different time scales, simultaneously. Stiffness is a challenging property that prevents conventional explicit numerical integrators from handling such problems efficiently, as the stability requirements dictate the choice of very small time-step. To overcome many of such limitations of conventional methods, there has been a renewed interest in using exponential integrator methods [2–5]. In this paper, we mainly focus on developing an efficient numerical scheme based on an exponential integrator method for the solution of some stiff stochastic fractional PDEs.

Fractional differential equations have attracted increasing attention in recent years, because they have applications in various fields of science and engineering. There are several definitions of a fractional derivative of order $\alpha > 0$ [6].

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The two most commonly used are the Riemann–Liouville and Caputo. The difference between the two definitions is in the way in which integrals and derivatives are interchanged. The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order, which unifies and generalizes the notions of integer-order differentiation and n-fold integration; for a history and developments of this field see, for example, [7,6,8]. Furthermore, fractional order differential equations have recently been shown to be valuable tools for modelling many phenomena in engineering, physics, fluid mechanics, viscoelasticity, mathematical biology, electrochemistry and other scientific fields. In spite of the fact that a considerable amount of research has been carried out on the theoretical analysis of these equations, analytic solution of many fractional differential equations cannot be obtained explicitly. So, many authors have resorted to numerical solution strategies, based on convergence and stability analysis [9–14].

Obviously, the need to use different stochastic perturbations, to model many natural phenomena, is unavoidable, and clearly the deterministic models cannot conquer uncertainty in model parameters. However, a suitable stochastic model has the ability to capture uncertainties explicitly. So, many phenomena in science and engineering have been modelled by stochastic partial differential equations (SPDEs) and also fractional stochastic partial differential equations. Among many research articles investigating the solution of SPDEs, analytically and numerically, we refer to a few recent ones; for instance, see [15–18].

Recently, there has been growing interest in the study of the fractional stochastic partial differential equations. There are many articles that investigated the effect of uncertainty and values of fractional order of derivative in the performance of solution of such equations, analytically and numerically. For example, [19] discussed numerical solution of stochastic fractional equations. The authors of [20] studied the nonlinear stochastic time fractional diffusion equations in the spatial domain $\mathbb{R}$, driven by multiplicative space–time white noise. Existence and uniqueness of random field solutions with measure-valued initial data, such as the Dirac delta measure, were established. Upper bounds on all pth moments, for $(p \geq 2)$, expressed by using a kernel function, were obtained. For some extension of the mentioned equations, the proof of existence and uniqueness of solution together with the moment bounds of the solution has been studied in [21,22]. The authors of [23] considered the non-linear space–time fractional stochastic heat type equations $\partial_t^{\beta} u_t(x) = -\upsilon(-\Delta)^{\alpha/2} u_t(x) + I_t^{1-\beta}\left[\lambda\sigma(u)\dot{F}(t,x)\right]$ in $(d+1)$ dimensions. They proved that absolute moments of the solutions of this equation grow exponentially; and the distances to the origin of the farthest high peaks of those moments grow exactly linearly with time. Existence and uniqueness of mild solutions to the above-mentioned equation were studied in [24]. The authors established conditions under which the solution is continuous, and under faster than linear growth of σ, they showed that time fractional stochastic partial differential equation has no finite energy solution. Moreover, under suitable conditions on the initial function, the asymptotic behaviour of the solution was studied in [25].

In this paper, we study the numerical solution of following stochastic time fractional (STF) PDEs with additive noise:

STF Burgers equation,

$$ {}_a^cD_t^{\alpha} u(x,t) + uu_x + \epsilon u_{xx} + \sigma B(t) = 0, \quad (x,t) \in \overline{\Omega}, \tag{1.1} $$

and STF KS equation,

$$ {}_a^cD_t^{\alpha} u(x,t) + uu_x + u_{xx} + u_{xxxx} + \sigma B(t) = 0, \quad (x,t) \in \overline{\Omega}. \tag{1.2} $$

For Eqs. (1.1) and (1.2), the boundary conditions are periodic, $\overline{\Omega} = \{(x,t)\,|\,x_L \leqslant x \leqslant x_R,\ 0 \leqslant t \leqslant T\}$, σ is a constant that indicates the amplitude of noise, $B(t)$ is a Wiener process on $L^2(\mathbb{R})$ (the space of real valued square integrable functions on $\mathbb{R}$), and ${}_a^cD_t^{\alpha}$ is the Caputo fractional derivative of order α defined by

$$ {}_0^cD_t^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^{\alpha}}, \quad \alpha \in (0,1). \tag{1.3} $$

We discuss the numerical PC expansion method, pioneered by Norbert Wiener [26], for the solution of STF equations (1.1) and (1.2). Through the PC expansion method the time fractional SPDE is reduced to a deterministic system of time fractional PDEs, enabling us to implement an appropriate deterministic method. Evaluation of some statistical quantities of the numerical solution of the time fractional SPDEs and the CPU time of solving such equations through the PC expansion method will confirm potential features of this method from the computational cost and precision points of view. Such efficient performance, in contrast to pure stochastic simulation, could be due to a better convergence rate of the deterministic methods employed in this paper. For discretization of the spatial part in the equation we use the Fourier spectral method. Taking advantage of using FFT through our programming code, improves the efficiency of simulation as well. Moreover, as mentioned before, to overcome stiffness limitations on time-step, we apply an exponential integrator for the time fractional variable discretization.

The organization of the paper is as follows: in Section 2, we explain the polynomial chaos expansion with some analytical investigation and some property of the Brownian motion. In Section 3, we briefly introduce the Fourier spectral method for space discretization and a fractional exponential integrator for the solution of system of fractional ordinary differential equations. Numerical computations, with some details, will be presented in Section 4 to illustrate the comparative efficiency of the method discussed in this paper. Finally, a brief conclusion is given in Section 5.

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