ARTICLE IN PRESS

Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

An inverse source problem for a two-parameter anomalous diffusion with local time datum

Khaled M. Furati [a,*], Olaniyi S. Iyiola [b], Kassem Mustapha [a]

[a] King Fahd University of Petroleum & Minerals, Department of Mathematics & Statistics, Dhahran 31261, Saudi Arabia

[b] University of Wisconsin–Milwaukee, Department of Mathematical Sciences, Milwaukee, WI, USA

ARTICLE INFO

Article history:
Available online xxxx

Keywords:
Inverse source problem
Fractional diffusion
Mittag-Leffler function
Hilfer derivative

ABSTRACT

We determine the space-dependent source term for a two-parameter fractional diffusion problem subject to nonlocal non-self-adjoint boundary conditions and two local time-distinct datum. A bi-orthogonal pair of bases is used to construct a series representation of the solution and the source term. The two local time conditions spare us from measuring the fractional integral initial conditions commonly associated with fractional derivatives. On the other hand, they lead to delicate 2×2 linear systems for the Fourier coefficients of the source term and of the fractional integral of the solution at $t = 0$. The asymptotic behavior and estimates of the generalized Mittag-Leffler function are used to establish the solvability of these linear systems, and to obtain sufficient conditions for the existence of our construction. Analytical and numerical examples are provided.

1. Introduction

Non-integer calculus and fractional differential equations have become an intrinsic tool in modeling different phenomena in many areas such as nanotechnology [1], control theory of dynamical systems [2,3], viscoelasticity [4], anomalous transport and anomalous diffusion [5], random walk [6,7], financial modeling [8] and biological modeling [9]. Some other physical and engineering processes are given in [10,11] and more applications can be found in the surveys in [12–14]. In particular, fractional models are increasingly adopted for processes with anomalous diffusion [15–17]. The featured role of the fractional derivatives is mainly due to their non-locality nature which is an intrinsic property of many complex systems [18].

In this paper, we consider determining the solution u and the space-dependent source f for the following two-parameter fractional diffusion equation (FDE):

$$\begin{aligned} &D^{\alpha,\gamma} u(x,t) - u_{xx}(x,t) = f(x), \quad x \in (0,1),\ t \in (0,T),\ 0 < \alpha \le \gamma \le 1, \\ &u(x,T_m) = z(x), \qquad u(x,T) = h(x), \quad x \in [0,1],\ 0 < T_m < T, \\ &u(1,t) = 0, \qquad u_x(0,t) = u_x(1,t), \quad t \in (0,T], \end{aligned} \tag{1}$$

where z and h are square integrable functions. The operator $D^{\alpha,\gamma}$ is the generalized Hilfer-type fractional derivative defined by

$$D^{\alpha,\gamma} y(t) = D^{\alpha}\left[y(t) - \frac{I^{1-\gamma} y(0)}{\Gamma(\gamma)}\, t^{\gamma-1} \right], \tag{2}$$

* Corresponding author.
E-mail addresses: kmfurati@kfupm.edu.sa (K.M. Furati), osiyiola@uwm.edu (O.S. Iyiola), kassem@kfupm.edu.sa (K. Mustapha).

http://dx.doi.org/10.1016/j.camwa.2016.06.036

where

$$I^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}y(\tau)\,d\tau,$$

and

$$D^{\alpha}y(t) = DI^{1-\alpha}y(t), \quad D = \frac{d}{dt},$$

are the Riemann–Liouville fractional integral and derivative, respectively. Noting that, when $I^{1-\gamma}y \in AC[0,T]$, then $D^{\alpha,\gamma}y$ can be written as

$$D^{\alpha,\gamma}y(t) = I^{\gamma-\alpha}D^{\gamma}y(t) = I^{\gamma-\alpha}DI^{1-\gamma}y(t). \tag{3}$$

Hence, $D^{\alpha,\gamma}$ reduces to the derivative introduced by Hilfer in [19] when $\gamma = \beta(1-\alpha)+\alpha$ with $0 < \beta \le 1$.

Observe that the Riemann–Liouville fractional derivative D^{α} and the Caputo fractional derivative ${}^cD^{\alpha} := I^{1-\alpha}D$ are special cases of the two-parameter fractional derivative $D^{\alpha,\gamma}$ for $\gamma = \alpha$ and $\gamma = 1$, respectively. Thus, $D^{\alpha,\gamma}$ is considered as an interpolant between D^{α} and ${}^cD^{\alpha}$.

The generalized two-parameter fractional source-free diffusion equation was proposed by Hilfer in [19]. A similar inverse source problem to (1) has been considered by Kirane et al. [20] but with the Caputo derivative for which the initial condition is the traditional local condition.

Furati et al. [21] constructed a series representation of u and f for problem (1), but subject to an integral-type initial condition instead, using a bi-orthogonal system. Unlike in [21], in the current problem, the value of the solution at some time $t = T_m$ is used rather than the value of the fractional integral of the solution $I^{1-\gamma}u(x,t)$ at $t = 0$, which may neither be measurable nor have a physical meaning. As a result, the construction of the series representation of u and f is not a straightforward extension of the one in [21]. This is mainly due to the complexity in ensuring the solvability of the arising linear systems and in achieving the necessary lower and upper bounds for showing the convergence of the constructed series.

Inverse source problems for a one-parameter FDE with Caputo derivative have been investigated by many researchers under various initial, boundary and over determination conditions. For a space-dependent source f, Zhang and Xu [22] used Duhamel's principle and an extra boundary condition to uniquely determine f. Kirane and Malik [20] studied first a one-dimensional problem with non-local non-self-adjoint boundary conditions and subject to initial and final conditions. The results were extended to the two-dimensional problem by Kirane et al. [23]. Özkum et al. [24] used Adomian decomposition method to construct the source term for a linear FDE with a variable coefficient in the half plane. In a bounded interval, Wang et al. [25] reconstructed a source for an ill posed time-FDE by Tikhonov regularization method. A numerical method for reproducing kernel Hilbert space to solve an inverse source problem for a two-dimensional problem is proposed by Wang et al. [26]. Wie and Wang [27] proposed a modified version of quasi-boundary value method to determine the source term in a bounded domain from a noisy final data. Analytic Fredholm theorem and some operator properties are used by Tatar and Ulusoy [28] to prove the well-posedness of a one-dimensional inverse source problem for a space–time FDE. Feng and Karimov [29] used eigenfunctions to analyze an inverse source problem for a fractional mixed parabolic hyperbolic equation. They formulated the problem into an optimization problem and then used semismooth Newton algorithm to solve it. Gülkaç [30] applied the Homotopy Perturbation Method to find the source term allowing space dependent diffusivity. For a three-dimensional inverse source problem, we refer the reader to the work by Sakamoto and Yamamoto [31] and by Ruan et al. [32]. In relation to above, for the case of time-dependent source term f, see [33–38].

In all the previous works cited above, the problems considered involve the Caputo derivative together with the classical initial conditions. In our work, we consider a two parameter fractional derivative, of which, the Caputo and Riemann–Liouville derivatives are special cases. As a result, we need to consider the possibility of having a nonlocal initial condition, which in general may not have a clear physical meaning or a direct way of measuring. We show that these nonlocal initial conditions can be replaced by a local observation.

The rest of the paper is organized as follows. In Section 2, we present a brief discussion of the generalized Mittag-Leffler function and derive some related properties. In addition, we establish the solvability of a 2×2 linear system with a coefficient matrix involving Mittag-Leffler functions and obtain estimates of these coefficients. We construct the series representations of the solution and source term in Section 3. The well-posedness of this construction is proved in Section 4. In Section 5, analytical and computational examples are presented.

2. Generalized Mittag-Leffler function

The Prabhakar generalized Mittag-Leffler function [39] is defined as

$$E^{\rho}_{\alpha,\beta}(w) = \sum_{k=0}^{\infty}\frac{\Gamma(\rho+k)}{\Gamma(\rho)\,\Gamma(\alpha k+\beta)}\frac{w^k}{k!}, \quad w,\beta,\rho \in \mathbb{C},\ \operatorname{Re}\alpha > 0. \tag{4}$$

Please cite this article in press as: K.M. Furati, et al., An inverse source problem for a two-parameter anomalous diffusion with local time datum, Computers and Mathematics with Applications (2016), http://dx.doi.org/10.1016/j.camwa.2016.06.036