ARTICLE IN PRESS

Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Weak solutions of the time-fractional Navier–Stokes equations and optimal control☆

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ARTICLE INFO

Article history:
Available online xxxx

Keywords:
Navier–Stokes equations
Caputo fractional derivative
Weak solutions
Existence
Optimal controls

ABSTRACT

In this paper, we deal with the Navier–Stokes equations with the time-fractional derivative of order $\alpha \in (0, 1)$, which can be used to simulate anomalous diffusion in fractal media. We firstly give the concept of the weak solutions and establish the existence criterion of weak solutions by means of Galerkin approximations in the case that the dimension $n \leq 4$. Moreover, a complete proof of the uniqueness is given when $n = 2$. At last we give a sufficient condition of optimal control pairs.

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1. Introduction

The Navier–Stokes equations which provide a natural description of the interaction of a viscous liquid with rigid bodies, are taken as important mathematical tools for the better acquainted with several actual problems in many important phenomena, such as thermo-hydraulics, aeronautical sciences, meteorology, the petroleum industry, plasma physics and so on. The motivation for the study of this equation comes from the increasing effort that mathematicians have devoted to these problems; see Galdi [1], Lemarié-Rieusset [2], Lukaszewicz and Kalita [3]. In the last few years considerable process has been made in the existence, uniqueness and smoothness properties of weak solutions related to the Navier–Stokes equations; see, e.g., Barbu [4], Duchon and Robert [5], Feireisl et al. [6], Jia and Šverák [7], Jüngel [8], Vasseur and Yu [9], Zhou [10] and the references therein. It is worth mentioning that Lions [11] was the first to carry out the study that the Navier–Stokes equations have a weak solution with time-fractional derivative of order less than $\frac{1}{4}$ provided the space dimension is not further than four. After that, only a few research results on this subject have been achieved; for example, Zhang [12] proved that this type of equations has a weak solution whose time-fractional derivative of order is no more than $\frac{1}{2}$. So it is reasonable and significative to pose the time-fractional Navier–Stokes equations, which characterize the long memory processes, thus they can be used to model anomalous diffusion in fractal media.

It is well-known that the considerable interest in the study of fractional calculus; see, e.g., Herrmann [13] and Hilfer [14], especially fractional partial differential equations such as time-fractional Navier–Stokes equations, has been motivated by applications in different fields of science and engineering. Recently, significant development has been made in existence and uniqueness of solutions to time-fractional Navier–Stokes equations. El-Shahed et al. [15], Ganji et al. [16], Momani and Zaid [17] studied the analytical solutions of the time-fractional Navier–Stokes equations by different methods. Very recently,

☆ Project supported by National Natural Science Foundation of China (11271309).

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http://dx.doi.org/10.1016/j.camwa.2016.07.007
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Please cite this article in press as: Y. Zhou, L. Peng, Weak solutions of the time-fractional Navier–Stokes equations and optimal control, Computers and Mathematics with Applications (2016), http://dx.doi.org/10.1016/j.camwa.2016.07.007

Carvalho-Neto and Gabriela [18], Zhou and Peng [19] derived the existence and uniqueness of local and global mild solutions, the regularity of classical solution for the time-fractional Navier–Stokes equations.

However, it should be stressed that the weak solution theory on the time-fractional Navier–Stokes equations is not yet sufficiently elaborated in contrast with that on integer-order case. In fact, it is essential and considerable to investigate weak solution problems for the time-fractional Navier–Stokes equations. To the best of our knowledge, the weak solution of this type of equations has not been researched. Therefore, in this paper we study weak solutions of the following time-fractional Navier–Stokes equations in an open set $\Omega \subset \mathbb{R}^n$ $(n \leq 4)$ with smooth boundary $\partial\Omega$:

$$\begin{cases} \partial_t^\alpha u - \nu\Delta u + (u\cdot\nabla)u = -\nabla p + f, \quad x\in\Omega,\ t\in[0,T], \\ \nabla\cdot u = 0, \quad (t,x)\in[0,T]\times\Omega, \qquad u(t,x)=0, \quad (t,x)\in[0,T]\times\partial\Omega, \\ u(0,x)=a, \quad x\in\Omega, \end{cases} \tag{1.1}$$

where ∂_t^α is the Caputo fractional derivative of order $\alpha \in (0,1)$, $u = (u_1(t,x), u_2(t,x), \ldots, u_n(t,x))$ represents the velocity field at a point $x \in \Omega$ and time $t > 0$, $\nu > 0$ is the viscosity, $p = p(t,x)$ is the pressure, $f = (f_1(t,x), f_2(t,x), \ldots, f_n(t,x))$ is the external force and $a = (a_1(t,x), a_2(t,x), \ldots, a_n(t,x))$ is the initial velocity.

Then we proceed to consider the systems with control in $\Omega \subset \mathbb{R}^2$:

$$\begin{cases} \partial_t^\alpha u - \nu\Delta u + (u\cdot\nabla)u = -\nabla p + C_0 w + f, \quad x\in\Omega,\ t\in[0,T], \\ \nabla\cdot u = 0, \quad (t,x)\in[0,T]\times\Omega, \qquad u(t,x)=0, \quad (t,x)\in[0,T]\times\partial\Omega, \\ u(0,x)=a, \quad x\in\Omega, \end{cases} \tag{1.2}$$

where U is a real Hilbert space, $w : [0,T] \to U$ and the operator $C_0 : U \to (L^2(\Omega))^2$ is linear continuous. Since many good properties satisfied in inter-order differential equations are not generalized directly to fractional-order case, we found it more challenging in dealing with weak solutions of equation (1.1). As we all know, the main difficulty to study weak solutions is how to give an appropriate definition of weak solutions, introduce a suitable "work space" and establish the estimates of inequality by using Galerkin approximations.

We begin in Section 2 with some notations and definitions. Section 3 is devoted to the proof of the existence for weak solutions to equation (1.1), then proceed to study the uniqueness of the weak solution for such equations in $\mathbb{R}^2$. Finally, Section 4 is concerned with an existence result of the optimal control of systems (1.2).

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Assume that X is a Banach space. Let $\alpha \in (0,1]$ and $v : [0,T] \to X$. We define the left and right Riemann–Liouville integrals of v:

$${}_0I_t^\alpha v(t) = \int_0^t g_\alpha(t-s)v(s)ds, \qquad {}_tI_T^\alpha v(t) = \int_t^T g_\alpha(s-t)v(s)ds, \quad t>0,$$

provided the integrals are point-wise defined on $[0,\infty)$, where g_α denotes the Riemann–Liouville kernel

$$g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad t>0.$$

Further, ${}_0^CD_t^\alpha$ and ${}_tD_T^\alpha$ stand the left Caputo and right Riemann–Liouville fractional derivative operators of order α, respectively; they are defined by

$${}_0^CD_t^\alpha v(t) = \int_0^t g_{1-\alpha}(t-s)\frac{d}{ds}v(s)ds, \qquad {}_tD_T^\alpha v(t) = -\frac{d}{dt}\left(\int_t^T g_{1-\alpha}(s-t)v(s)ds\right), \quad t>0.$$

More generally, for $u : [0,\infty)\times\mathbb{R}^n \to \mathbb{R}^n$, the left Caputo fractional derivative with respect to time of the function u can be written as

$$\partial_t^\alpha u(t,x) = \int_0^t g_{1-\alpha}(t-s)\frac{\partial}{\partial_s}u(s,x)ds, \quad t>0.$$

Let $v : \mathbb{R} \to X$. We define the Liouville–Weyl fractional integral and the Caputo fractional derivative on the real axis:

$${}_{-\infty}I_t^\alpha v(t) = \int_{-\infty}^t g_\alpha(t-s)v(s)ds, \qquad {}_{-\infty}^{\ C}D_t^\alpha v(t) = \int_{-\infty}^t g_{1-\alpha}(t-s)\frac{d}{ds}v(s)ds,$$

respectively.

We also need for our purposes the fractional integration by parts in the formula (see [20]):

$$\int_0^T (\partial_t^\alpha u(t), \psi(t))dt = \int_0^T (u(t), {}_tD_T^\alpha\psi(t))dt + (u(t), {}_tI_T^{1-\alpha}\psi(t))|_0^T,$$

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