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Generalized distributed order diffusion equations with composite time fractional derivative

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ABSTRACT

In this paper we investigate the solution of generalized distributed order diffusion equations with composite time fractional derivative by using the Fourier–Laplace transform method. We represent solutions in terms of infinite series in Fox H-functions. The fractional and second moments are derived by using Mittag-Leffler functions. We observe decelerating anomalous subdiffusion in case of two composite time fractional derivatives. Generalized uniformly distributed order diffusion equation, as a model for strong anomalous behavior, is analyzed by using Tauberian theorem. Some previously obtained results are special cases of those presented in this paper.

1. Introduction

The fractional differential equations have attracted much attention due to their application in different fields of science [1–5]. Time fractional diffusion equations have been used to model anomalous diffusion processes in complex systems. They can be introduced either by using Caputo time fractional derivative or Riemann–Liouville (R–L) time fractional derivative. By using the Caputo time fractional derivative the corresponding equation which describes an anomalous diffusion process with transport exponent $0 < \mu < 1$ is given by [3] (see also Ref. [6])

$$\frac{\partial^{\mu}}{\partial t^{\mu}} W(x,t) = \mathcal{D}_{\mu} \frac{\partial^2}{\partial x^2} W(x,t), \tag{1}$$

where $W(x,t)$ is the probability distribution function (PDF),

$$\frac{\partial^{\mu}}{\partial t^{\mu}} f(t) = {}_{C}D_{t}^{\mu} f(t) = \frac{1}{\Gamma(1-\mu)} \int_0^t \frac{\frac{\mathrm{d}}{\mathrm{d}t'} f(t')}{(t-t')^{\mu}} \mathrm{d}t', \quad 0 < \mu < 1 \tag{2}$$

is the Caputo fractional derivative [7,8], and $\mathcal{D}_{\mu}$ is the generalized diffusion coefficient. Contrary to this, the R–L fractional derivative is given by [4,8]

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$$ {}_{RL}D_t^{\mu}f(t) = \frac{1}{\Gamma(1-\mu)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{f(t')}{(t-t')^{\mu}}\mathrm{d}t', \quad 0<\mu<1. $$

The initial condition is usually taken as $W(x,0+) = \delta(x)$, and the boundary conditions are given by $W(\pm\infty,t) = 0$, $\frac{\partial}{\partial x}W(\pm\infty,t) = 0$ in unbounded domain in space of Lebesgue integrable functions. These are natural boundary conditions which are used in order to have a unique solution of the equation, as well as the solution to be bounded and differentiable. Note that these conditions are also used in order that the Fourier transform ($\mathcal{F}[f(x)] = \int_{-\infty}^{\infty} f(x)e^{\imath kx}\mathrm{d}x$) of the second derivative to be given by $\mathcal{F}\left[f''(x)\right] = -|k|^2\mathcal{F}[f(x)]$. The fractional transport equation (1) is characterized by the single exponent μ, since the mean square displacement (MSD) is given by $\left\langle x^2(t)\right\rangle = \int_{-\infty}^{\infty} x^2 W(x,t)\mathrm{d}x = 2\mathcal{D}_{\mu}\frac{t^{\mu}}{\Gamma(1+\mu)}$ [3].

Time fractional diffusion equation with single fractional exponent can be generalized by the introduction of distributed order time fractional operators [9–11]. Thus, the distributed order diffusion equation is given by [9]

$$ \int_0^1 \tau^{\mu-1}p(\mu)\frac{\partial^{\mu}}{\partial t^{\mu}}W(x,t)\mathrm{d}\mu = \mathcal{D}\frac{\partial^2}{\partial x^2}W(x,t), \tag{3} $$

where the diffusion coefficient $\mathcal{D}$ has the conventional dimension m^2/s. Here, $\frac{\partial^{\mu}}{\partial t^{\mu}}$ is the Caputo time fractional derivative (2) of order $0<\mu<1$, $p(\mu)$ is a weight function, i.e., dimensionless non-negative function with

$$ \int_0^1 p(\mu)\mathrm{d}\mu = 1, \tag{4} $$

and τ is a time parameter with dimension $[\tau] = \mathrm{s}$.

Generalization of time fractional and distributed order time fractional diffusion equations can be done in the framework of the CTRW theory by introduction of generalized waiting time PDF [3] with a given non-negative integrable function, which appears as a memory kernel from the left hand side in the diffusion equation.

Such distributed order diffusion equations have been shown to represent useful tool for modeling decelerating anomalous diffusion, ultraslow diffusive processes and strong anomaly [9–19]. These equations have been recently shown to possess multiscaling properties [17,18].

Further generalization can be done by introduction of space fractional derivative instead of the second space derivative, for example, space fractional Riesz–Feller derivative of order α and skewness θ, which is defined as a pseudo-differential operator with a symbol $\psi_{\alpha}^{\theta}(k) = |k|^{\alpha}e^{\imath(\mathrm{sgn}k)\theta\pi/2}$, which is the logarithm of the characteristic function of a general Lévy strictly stable probability density with index of stability α and asymmetry parameter θ. Its Fourier transform is given by [20]

$$ \mathcal{F}\left[D_{\theta}^{\alpha}f(x)\right](k) = -\psi_{\alpha}^{\theta}(k)\,\mathcal{F}[f(x)](k). \tag{5} $$

If $\theta = 0$ then (5) reduces to

$$ \mathcal{F}\left[D_0^{\alpha}f(x)\right](k) = -|k|^{\alpha}\mathcal{F}[f(x)](k). \tag{6} $$

For $0<\alpha\le 2$ and $|\theta|\le\min\{\alpha, 2-\alpha\}$, the Riesz–Feller fractional derivative is given by [20]

$$ D_{\theta}^{\alpha}f(x) = \frac{\Gamma(1+\alpha)}{\pi}\left\{\sin((\alpha+\theta)\pi/2)\int_0^{\infty}\frac{f(x+\xi)-f(x)}{\xi^{1+\alpha}}\mathrm{d}\xi \right. $$
$$ \left. +\sin((\alpha-\theta)\pi/2)\int_0^{\infty}\frac{f(x-\xi)-f(x)}{\xi^{1+\alpha}}\mathrm{d}\xi\right\}. \tag{7} $$

The solution of the space fractional diffusion equations with space fractional Riesz–Feller derivative corresponds to the PDF obtained from the CTRW theory for Lévy flights [3].

In the present paper, we introduce a new generalization of distributed order time fractional diffusion equation (3), where instead of Caputo time fractional derivative we use Hilfer-composite time fractional derivative.

This paper is organized as follows. In Section 2 we formulate the problem of generalized distributed order diffusion equation, and we consider different forms of the weight function. We analyze the PDF and second moment by using the Fourier–Laplace transform method. We observe decelerating anomalous subdiffusion in the case of two composite time fractional derivatives. For uniformly distributed order diffusion equation we apply the Tauberian theorem in order to find the behavior of the second moment in the long and short time limit. Due to the contribution of all fractional exponents μ between zero and one, we observe ultraslow diffusion and strong anomaly. In Section 3 we consider generalized distributed order Fokker–Planck–Smoluchowski equation with two composite time fractional derivatives, and we analyze the relaxation of modes and the case of presence of an external harmonic potential. The results are exact and represented in terms of the Mittag-Leffler and Fox H-functions. The Summary is given in Section 4. Appendices are added at the end of the paper.

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