ARTICLE IN PRESS

Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Galerkin spectral method for the fractional nonlocal thermistor problem

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ARTICLE INFO

Article history:
Available online xxxx

Keywords:
Fractional differential equations
Finite difference method
Galerkin spectral method
Stability
Error analysis

ABSTRACT

We develop and analyse a numerical method for the time-fractional nonlocal thermistor problem. By rigorous proofs, some error estimates in different contexts are derived, showing that the combination of the backward differentiation in time and the Galerkin spectral method in space leads, for an enough smooth solution, to an approximation of exponential convergence in space.

1. Introduction

Fractional derivatives express properties of memory and heredity of materials, which is their main benefit when compared with integer-order derivatives. Practical problems require definitions of fractional derivatives that allow the use of physically interpretable initial conditions. Fractional time derivatives are linked with irregular sub-diffusion, where a darken of particles spread slower than in classical diffusion. The fractional space derivatives are used to model irregular diffusion or dispersion, where a particle spreads at a rate that does not agree with the classical Brownian motion, and the following can be asymmetric [1].

Fractional differential and integro-differential equations occur in different real processes and physical phenomena, such as in signal processing and image processing, optics, engineering, control theory, computer science (such as real neural networks, complex neural networks and information technology), statistics and probability, astronomy, geophysics, hydrology, chemical technology, materials, robots, earthquake analysis, electric fractal network, statistical mechanics, biotechnology, medicine, and economics [2–5].

In this paper, we consider the problem of the nonlocal time-fractional thermistor problem. This fractional model is obtained from the integer order one

$$\frac{\partial u(x,t)}{\partial t} - \triangle u = \frac{\lambda f(u)}{\left(\int_\Omega f(u)\,dx\right)^2}, \quad \text{in } Q_T = \Omega \times (0,T), \tag{1}$$

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http://dx.doi.org/10.1016/j.camwa.2016.05.033
0898-1221/

Please cite this article in press as: M.R. Sidi Ammi, D.F.M. Torres, Galerkin spectral method for the fractional nonlocal thermistor problem, Computers and Mathematics with Applications (2016), http://dx.doi.org/10.1016/j.camwa.2016.05.033

by replacing the derivative term by a fractional derivative of order $\alpha > 0$:

$$\begin{aligned} &\frac{\partial^\alpha u}{\partial t^\alpha} - \triangle u = \frac{\lambda f(u)}{\left(\int_\Omega f(u)\,dx\right)^2}, \quad \text{in } Q_T = \Omega \times (0,T), \\ &\frac{\partial u}{\partial n} = 0, \quad \text{on } S_T = \partial\Omega \times (0,T), \\ &u(0) = u_0, \quad \text{in } \Omega, \end{aligned} \tag{2}$$

where $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ denotes the Caputo fractional derivative of order α, $0 < \alpha < 1$, as defined in [6] and given by

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\alpha}, \quad 0 < \alpha < 1,$$

with $\triangle$ the Laplacian with respect to the spacial variables and where f is assumed to be a smooth function, as prescribed below, and T is a fixed positive real. Here n denotes the outward unit normal and $\frac{\partial}{\partial n} = n \cdot \nabla$ is the normal derivative on $\partial\Omega$. Such problems arise in many applications, for instance, in studying the heat transfer in a resistor device whose electrical conductivity f is strongly dependent on the temperature u. Constant λ is a dimensionless parameter, which can be identified with the square of the applied potential difference at the ends of the conductor. Function u represents the temperature generated by the electric current flowing through a conductor.

A fractional order model instead of its classical integer order counterpart has been considered here because fractional order differential equations are generalizations of integer order differential equations and fractional order models possess memory. Moreover, the fact that resistors are influenced by memory makes fractional modelling appropriate for this kind of dynamical problems. We use Caputo's definition. The main advantage is that the initial conditions for fractional differential equations with Caputo derivatives take the same form as for integer-order differential equations. Note that (2) covers (1) and extends it to more general cases. The classical nonlocal thermistor problem (1) with the time derivative of integer order can be obtained by taking the limit $\alpha \to 1$ in (2) (see [7]), while the case $\alpha = 0$ corresponds to the steady state thermistor problem. In the case $0 < \alpha < 1$, the Caputo fractional derivative depends on and uses the information of the solutions at all previous time levels (non-Markovian process). In this case the physical interpretation of fractional derivative is that it represents a degree of memory in the diffusing material. Such kind of models have been analytically investigated by a number of authors, using Green functions, the Laplace and Fourier–Laplace transform methods, in order to construct analytical solutions. However, papers in the literature on the numerical solutions of time fractional differential equations are still under development. In [8], existence and uniqueness of a positive solution to a generalized spatial fractional-order nonlocal thermistor problem is proved. Stability and error analysis of the semi-discretized fractional nonlocal thermistor problem is investigated in [9,10]. More precisely, in [9,10] a finite difference method is proposed, respectively for solving the semidiscretized fractional nonlocal thermistor problem and the time fractional thermistor problem, which is a system of elliptic–parabolic PDEs and where some stability as well as error analysis for this scheme is derived for both problems. Herein, an approach based on finite differences combined with the Galerkin spectral method is used to solve the nonlocal time fractional thermistor problem. By definition of fractional derivative, to compute the solution at the current time level one needs to save all the previous solutions, which makes the storage expensive if low-order methods are employed for spatial discretization. One of the main advantage of the spectral method is the fact that it can relax this storage limit since it needs fewer grid points to produce a highly accurate solution [11,12].

The text is organized as follows. In Section 2 a finite difference scheme for the temporal discretization of problem (2) is introduced. Then, in Section 3, we provide a finite difference-Galerkin spectral method to obtain error estimates of $(2-\alpha)$-order convergence in time and exponential convergence in space, for smooth enough solutions. The proof of our main result (Theorem 3) is given in Section 4. Finally, in Section 5 we carry out an error analysis between the solution u_N^k of the full discretized problem and the exact solution u. We end with Section 6 of conclusions and future work.

2. Time discretization: a finite difference scheme

Several theoretical analyses, on various aspects of both steady-state and time-dependent thermistor equations, with different aspects and types of boundary and initial conditions, have been carried out in the literature. For existence of weak solutions, uniqueness and related regularity and smoothness results, in several settings and under different assumptions on the coefficients, we refer the reader to [13]. For our purposes, the L^∞-energy method is a suitable and powerful tool to prove existence, regularity, and uniqueness of solutions to (2). From the results of [14], it follows by the L^∞-energy method that problem (2) has a unique and sufficiently smooth solution under the following assumptions:

(H1) $f : \mathbb{R} \to \mathbb{R}$ is a positive Lipschitz and $\mathbb{C}^1$ continuous function;
(H2) there exist positive constants c and β such that for all $\xi \in \mathbb{R}$ we have $c \le f(\xi) \le c|\xi|^{\beta+1} + c$;
(H3) $u_0 \in W^{1,\infty}(\Omega)$.

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