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Convergence and superconvergence of a fully-discrete scheme for multi-term time fractional diffusion equations

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ABSTRACT

Using finite element method in spatial direction and classical $L1$ approximation in temporal direction, a fully-discrete scheme is established for a class of two-dimensional multi-term time fractional diffusion equations with Caputo fractional derivatives. The stability analysis of the approximate scheme is proposed. The spatial global superconvergence and temporal convergence of order $O(h^2 + \tau^{2-\alpha})$ for the original variable in H^1-norm is presented by means of properties of bilinear element and interpolation postprocessing technique, where h and τ are the step sizes in space and time, respectively. Finally, several numerical examples are implemented to evaluate the efficiency of the theoretical results.

1. Introduction

Fractional partial differential equations (FPDEs) have become increasingly popular and important in many scientific and engineering fields. Compared with integer-order PDEs, FPDEs are better choices for describing some phenomena or processes with memory, hereditary and long-range interaction in viscoelasticity, diffusion, biology, relaxation vibrations, electrochemistry, finance and fluid mechanics [1–12]. Moreover, some underlying processes can be more accurately and flexibly modeled by multi-term FPDEs. For example, the multi-term time fractional diffusion-wave equation is a satisfying mathematical model for viscoelastic damping (see [13]). In [14], a two-term fractional diffusion equation has been successfully used for distinguishing different states in solute transport. Further investigations and applications of multi-term FPDEs can be found in [10,15–27].

In this paper, using the spatial finite element method and classical $L1$ approximation, we study a fully-discrete approximation scheme for the following two-dimensional multi-term time fractional diffusion equation:

$$\begin{cases} P(D_t)u(\mathbf{x},t) - \Delta u(\mathbf{x},t) = f(\mathbf{x},t), & (\mathbf{x},t) \in \Omega \times (0,T], \\ u(\mathbf{x},t) = 0, & (\mathbf{x},t) \in \partial\Omega \times (0,T], \\ u(\mathbf{x},0) = u_0(\mathbf{x}), & \mathbf{x} \in \Omega, \end{cases} \tag{1}$$

where $\Omega \subset R^2$ is a bounded convex polygonal region with boundary $\partial\Omega$, $\mathbf{x} = (x, y)$, $u_0(\mathbf{x})$ and $f(\mathbf{x}, t)$ are given functions assumed to be sufficiently smooth, and the operator $P(D_t)$ is defined by

$$P(D_t) = D_t^{\alpha} + \sum_{i=1}^{r} l_i D_t^{\alpha_i}, \quad l_i > 0,\ r \in N^+,\ 0 < \alpha_1 < \alpha_2 \cdots < \alpha_r < \alpha \leq 1,$$

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and D_t^β is the left-sided Caputo fractional derivative of order β with respect to t as defined in [2]:

$$D_t^\beta u(\mathbf{x},t)=\frac{1}{\Gamma(1-\beta)}\int_0^t \frac{\partial u(\mathbf{x},s)}{\partial s}\frac{ds}{(t-s)^\beta},\quad 0<\beta<1,$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Some fully-discrete approximate schemes based on finite element methods (FEMs) and the V-cycle multigrid method were presented for multi-term time–space fractional advection diffusion equations and the corresponding unconditional stability analysis and convergence results were deduced in [15]. Analytical solutions were studied for multi-term time–space fractional reaction–diffusion equations on an infinite domain in [16]. Liu et al. derived analytical solutions for the multi-term time–space Caputo–Riesz fractional advection–diffusion equations on a finite domain (see [17]). An efficient operational formulation of spectral tau method for multi-term time–space fractional differential equation with Dirichlet boundary conditions was proposed in [18]. With regard to multi-term time FPDEs, FEMs, spectral methods and various finite difference methods (FDMs) are commonly used. For example, Liu et al. presented numerical approximations for multi-term time-fractional diffusion equations by using spectral method in [19] and for multi-term time-fractional wave equations by means of FDMs in [20], respectively. [21] discussed the stability and convergence of a fully-discrete scheme for a one-dimensional multi-term time FPDE by spatial FEM and Diethelm fractional backward difference method in the temporal direction. In [22], a space finite element semi-discrete scheme and a stable fully-discrete scheme were proposed for (1) and nearly optimal error estimates were provided for both cases of smooth and nonsmooth initial data and inhomogeneous term. Some fully-discrete schemes for one- and two-dimensional multi-term time-fractional sub-diffusion equations were proposed by combining the compact difference method for spatial discretization and $L1$ approximation for multi-term time Caputo fractional derivatives in [23]. Some difference approximate schemes were presented for the multi-term time-fractional diffusion-wave equation and the corresponding unconditional stability and global convergence were provided in [24]. In addition, there are some studies about mathematical analysis for multi-term time FPDEs. For example, the well-posedness and the long-time asymptotic behavior of the solution for the initial–boundary value problem (1) were investigated in [25]. The initial–boundary value problems for the generalized multi-term time-fractional diffusion equation with variable coefficients were considered in [26]. Analytical solutions for the multi-term time-fractional diffusion-wave/diffusion equations were presented by a method of separating variables in [27].

Based on advantages of the spatial finite element method and classical $L1$ approximation, the goal of this paper is to develop an unconditionally stable fully-discrete scheme for (1) and obtain global superconvergence results without Ritz projection and the restrictions between h and τ, where h, τ are spatial size and time step, respectively. For single-term time FPDEs, there are a few works on superconvergence analysis. For example, Mustapha et al. [28] constructed a fully-discrete scheme with a spatial hybridizable discontinuous Galerkin (HDG) method and a temporal generalized Crank–Nicolson scheme, which resulted in a superconvergence with order h^{k+2} for $k\geq 1$ by means of same projection in [29,30], where k, h were the degree of piecewise polynomials of HDG approximations and the maximum diameter of the elements of the mesh, respectively. In [31], a numerical approximate scheme was established by combining a piecewise-linear DG method in time and a piecewise-linear conforming FEM in space with a second-order nodal superconvergence result in space. However, to the authors' knowledge, published works on global superconvergence analysis of the Galerkin FEMs for multi-term time FPDEs of the form (1) are quite limited.

As is known, superconvergence is an efficient procedure for improving approximate accuracy of FEMs [32–42]. By use of the properties of integral identities, Lin et al. have developed a novel superconvergence theory for FEMs (see [33,34]). Regarding more works on superconvergence analysis of FEMs for various integer-order time–space PDEs, we refer to [38–42]. In addition, a brief introduction for $L1$ approximation is given in [11,12,43]. $L1$ approximation is commonly used to discretize the Caputo fractional derivative of order $\alpha(0<\alpha<1)$ with accuracy of order $2-\alpha$ (see, for example, [22,23,44–47]).

The remainder of the paper is organized as follows. In Section 2, a fully-discrete scheme is proposed for (1) based on bilinear FEM and $L1$ approximation. Some necessary lemmas for the unconditional stability analysis and error estimates of the proposed scheme are given in Section 3. In Section 4, the unconditional stability analysis is shown; moreover, the corresponding convergence results in L^2-norm and superconvergence in H^1-norm are deduced. Several numerical examples are implemented in Section 5, which tests the efficiency of the theoretical results. In Section 6, some conclusions are drawn.

2. FEM fully-discrete approximate scheme

Let Γ_h be a family of rectangular meshes of Ω with $\bar{\Omega}=\cup_{K\in\Gamma_h}K$. For each $K\in\Gamma_h$, assume that O_K is the center of K, where $O_K=(x_K,y_K)$ and $h_{x,K}$, $h_{y,K}$ be the perpendicular distances between O_K and two sides of K which are parallel to the two coordinate planes, respectively. The four vertices of K are denoted by $d_1=(x_K-h_{x,K},y_K-h_{y,K})$, $d_2=(x_K+h_{x,K},y_K-h_{y,K})$, $d_3=(x_K+h_{x,K},y_K+h_{y,K})$, $d_4=(x_K-h_{x,K},y_K+h_{y,K})$. Assume that $h_K=\max\{h_{x,K},h_{y,K}\}$, $h=\max_{K\in\Gamma_h}h_K$.

Define the finite element spaces by

$$V_h=\{v;\ v|_K\in Q_{11}(K)\},\qquad V_0^h=\{v\in V_h;\ v|_{\partial\Omega}=0\},$$

where $Q_{ij}(K)=\mathrm{span}\{x^r y^s|\ (x,y)\in K,\ 0\leq r\leq i, 0\leq s\leq j\}$.

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