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Simulation of the approximate solutions of the time-fractional multi-term wave equations

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ABSTRACT

In this paper, simulations of the approximation solutions of time-fractional wave, forced wave (shear wave), and damped wave equations are given. The common finite difference rules besides the backward Grünwald–Letnikov scheme are used to find the approximation solution of these models. The paper discusses also the effects of the memory, the internal force (resistance) and the external force on the travelling wave. The Von-Neumann stability conditions are also considered and discussed for these models. Besides the simulations of the time evolutions of the approximation solutions, the stationary solutions are also simulated. The numerical results are obtained by the Mathematica software.

1. Introduction

The time-fractional multi-term wave equations are linear integro differential equations. They are obtained from their corresponding classical multi-term wave equations by replacing the second-order time derivative by fractional derivative of order $1 < \beta \leq 2$. There are a growing interests in studying these equations because of their importance in modelling many physical, biological, medical, chemical and many other fields. For example, the over diagnostic ultrasound frequencies, acoustic absorption in biological tissue exhibits a power law with a non integer frequency, see [1,2]. Also in a complex inhomogeneous conducting medium experimental evidence shows that the sound waves propagate with power law of non integer order. For further applications on physics and on real phenomena, see [3–5]. The Caputo time fractional operator has been widely used instead of the second time derivative to model mathematically such problems in order to discuss the effect of the memory on the studied system, see [6]. The time-fractional wave equation reads

$$ {}_{t}D_{*}^{\beta} u(x,t) = a\frac{\partial^2 u}{\partial x^2}, \quad a > 0,\ 1 < \beta \leq 2. \tag{1.1}$$

Many mathematicians prefer to call it time-fractional diffusion wave equation. This equation has been treated by many authors. Most of these treatments are based on the iterative methods which give hardly approximation solutions which are not valid on the long run. Stojanovic, [7], used an approximation method based on Laguerre polynomials to find a numerical solution to diffusion wave phenomena. Agrawal, [8] discussed the general solution for the fourth-order fractional diffusion-wave equation. Luchko [9] has obtained the fundamental solution of the multi-dimensional fractional wave equation and obtained the solutions of many special cases in the form of convergent series. Gorenflo [10] has discussed the stochastic processes related to the fractional wave equations and its distributed order.

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Adding an attractive force at the LHS of Eq. (1.1), causes it to model a wide range of physical phenomena. For example the motion of the internal waves which occur throughout the atmosphere and oceans under the action of external waves such as pressure and the thermodynamic effects. The mathematical equation that models this phenomenon reads

$$ {}_{t}D_{*}^{\beta}u(x,t) = a\frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x}(F(x)u(x,t)), \quad a > 0, \tag{1.2}$$

where $F(x)$ is the external force. This equation is a special form of the multi-term time-fractional differential equations which increasingly begin to receive the attention of a huge number of authors. For instance, in the papers [11,12] a two-term time fractional differential equation, which includes a concrete case of fractional diffusion-wave problem, is studied in the abstract context. If the propagation of the wave is under the action of an internal force, then it reads

$$ {}_{t}D_{*}^{\beta}u(x,t) = k\frac{\partial u}{\partial t} + a\frac{\partial^2 u}{\partial x^2}, \quad a > 0,\ k > 0. \tag{1.3}$$

Some physicians prefer to call this equation the time-fractional telegraph equation which governs the electrical transmission in a telegraph cable. In this paper, we prefer to call it time-fractional damped wave equation. Our aim in this paper is to find the approximation solutions of the above equations by using the finite difference scheme and to compare the time evolution of them. The comparison includes the studies of the external, internal forces besides the effect of the memory on the propagation of the waves. The stability test is also studied for each model.

Therefore, the paper is organized as follows: Section 1 is devoted to the introduction. Section 2, is devoted to introduce the discretization of the classical, forced and damped wave equations by using the common finite difference rules. In Section 3, we discuss the finite difference scheme of the time fractional wave, forced wave, and damped wave equations. In Section 4, we study the stability necessary conditions. Section 5, is devoted to simulate the wave propagation of the previous models for different values of the time order β and to interpret our numerical results with investigating to the effect of the memory on the propagation of the waves.

2. Discretization of the classical wave, forced wave, and damped wave equations

The classical wave equation models many physical phenomena such as wave propagation of sound travelling in a fluid, in gas, or in any other ideal medium. It models also an electrical signal travelling along transmission cable. The traditional wave equation models a vibrating string of length $L = 2R + 1$ being fixed at both its ends and reads

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a\frac{\partial^2 u(x,t)}{\partial x^2}, \quad -R \le x \le R,\ t > 0. \tag{2.1}$$

Here $u(x, t)$ is the vertical displacement of the vibrating string and is imposed to the Dirichlet boundary conditions

$$u(x,0) = f(x) \tag{2.2}$$
$$u_t(x,0) = 0 \tag{2.3}$$
$$u(-R,t) = u(R,t) = 0, \tag{2.4}$$

here $a > 0$ is the general positive constant. To find the approximation solution of the classical wave equation (2.1), we utilize the symmetric difference in space forward in time. The space variable x is discretized by the grid points

$$x_j = jh, \quad h > 0,\ j \in \mathbb{Z}, \tag{2.5}$$

where, j is restricted to the interval $j \in [-R, R]$, $h = 1/(2R + 1)$, where $R \in \mathbb{Z}$. Adjust τ as

$$t_n = n\tau, \quad \tau > 0,\ n \in \mathbb{N}_0. \tag{2.6}$$

Introduce the clump $y_j^{(n)}$ to approximate the integral of the displacement function $u(x, t)$ over the small interval h. Then constitute the vector $y^{(n)} = \{y_{-R}^{(n)}, y_{-R+1}^{(n)}, \dots, y_{R-1}^{(n)}, y_R^{(n)}\}^T$, with a suitably initial value $y^{(0)}$ being obtained by the aid of initial condition $u(x, 0) = f(x)$. Discretizing Eq. (2.1) and solving it for $y_j^{(n+1)}$, one gets

$$y_j^{(n+1)} = -y_j^{(n-1)} + (2 - 2a\mu)y_j^{(n)} + a\mu y_{j+1}^{(n)} + a\mu y_{j-1}^{(n)} + O(\tau^2, h^2). \tag{2.7}$$

This equation can be written in the matrix form as

$$y^{(n+1)} = -y^{(n-1)} + P^{\top} \cdot y^{(n)}, \tag{2.8}$$

where P is a tri-diagonal matrix defined as

$$P_{ij} = \begin{cases} P_{ij}^{(1)} = a\mu & j = i+1,\ i = -R, \dots, R-1 \\ P_{ij}^{(2)} = (2 - 2a\mu) & j = i = -R, \dots, R \\ P_{ij}^{(3)} = a\mu & j = i-1,\ i = -R+1, \dots, R-1, \end{cases} \tag{2.9}$$