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Efficient two-dimensional simulations of the fractional Szabo equation with different time-stepping schemes ☆

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ABSTRACT

The modified Szabo wave equation is one of the various models that have been developed to model the power law frequency-dependent attenuation phenomena in lossy media. The purpose of this study is to develop two different efficient numerical methods for the two-dimensional Szabo equation and to compare the relative merits of each method. In both methods we employ the ADI scheme to split directions, however, we use different time discretization. Specifically, in the first ADI method (ADI-I) we include a third-order correction term to achieve second-order convergence for smooth solutions, hence extending the work of Sun and Wu (2006). In the second ADI method (ADI-II), we employ the scheme in Zeng et al. (submitted for publication) to two dimensional fractional wave equation using multiple correction terms to enhance accuracy for non-smooth solutions. Our simulation results show that both methods are computationally efficient for the fractional wave equation but have different advantages in terms of accuracy. Specifically, ADI-II seems to produce more accurate results than ADI-I for non-smooth solutions. However, for smooth solutions and fractional order close to two, ADI-I seems to outperform ADI-II.

1. Introduction

The attenuation of wave energy in lossy media $\gamma(\cdot)$ exhibits a frequency dependency characterized by a power law [1–4]

$$\gamma(\omega) = \gamma_0 |\omega|^{\alpha}, \tag{1}$$

in which γ_0 and α are empirical parameters obtained by fitting measured data. It has been found that when the sound travels in lossy media, such as biological tissues and sediment, the exponent α falls between $(0, 2)$, as shown in Fig. 1 (i.e., α is the slope). The classical damped wave equation corresponds to $\alpha = 0$, which shows frequency independence. However, in real applications there is a frequency-squared dependent attenuation, which means $\alpha = 2$ [2,3].

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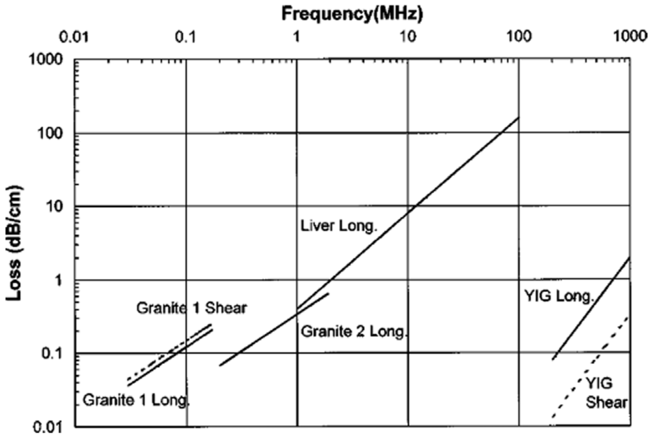

Fig. 1. Power law frequency-dependent wave attenuation of biological tissues.
Source: Adopted from Ref. [5].

Szabo [2,3] proposed the following equation written in the time-domain for both longitudinal and shear waves with an attenuation term of the form,

$$\frac{1}{c_0^2}\partial_t^2 u + \frac{2\alpha_0}{c_0} L_{t,\alpha}(u) = \Delta u, \tag{2}$$

where the operator $L_{t,\alpha}(\cdot)$ is defined as follows:

$$L_{t,\alpha}(u) = \begin{cases} \partial_t u, \quad \alpha = 0, \\ -\dfrac{2\Gamma(\alpha+2)\cos[(\alpha+1)\pi/2]}{\pi} \displaystyle\int_0^t \frac{u(\tau)}{(t-\tau)^{\alpha+2}} d\tau, \quad 0 < \alpha < 2, \\ \partial_t^3 u, \quad \alpha = 2, \end{cases} \tag{3}$$

and c_0 is the sound speed. However, this model contains a convolutional operator which brings in a hypersingularity. Thus, Chen and Holm [4] modified the model by incorporating a positive fractional derivative, which is formulated as

$$\frac{1}{c_0^2}\partial_t^2 u + \frac{2\alpha_0}{c_0} Q_{t,\alpha}(u) = \Delta u, \tag{4}$$

where the fractional operator $Q_{t,\alpha}(\cdot)$ instead of $L_{t,\alpha}(\cdot)$ is defined as follows:

$$Q_{t,\alpha}(u) = \begin{cases} \partial_t u, \quad \alpha = 0, \\ \partial_t^{\alpha+1} u, \quad 0 < \alpha < 2, \\ \partial_t^3 u, \quad \alpha = 2. \end{cases} \tag{5}$$

The positive fractional derivative is defined by

$$\partial_t^{\alpha+1} u(t) = \begin{cases} \dfrac{1}{(\alpha+1)\alpha q(\alpha+1)} \displaystyle\int_0^t \frac{u''(\tau)}{(t-\tau)^{\alpha}} d\tau, \quad 0 < \alpha \le 1, \\ \dfrac{-1}{(\alpha-1)q(\alpha+1)} \displaystyle\int_0^t \frac{u'''(\tau)}{(t-\tau)^{\alpha-1}} d\tau, \quad 1 < \alpha < 2, \end{cases} \tag{6}$$

where $q(\alpha)$ can be written as

$$q(\alpha) = \frac{\pi}{2\Gamma(\alpha+1)\cos[(\alpha+1)\pi/2]}. \tag{7}$$

Thus, the modified Szabo wave equation can also be written as

$$\frac{1}{c_0^2}\partial_t^2 u + \frac{2\alpha_0}{c_0\cos(\alpha\pi/2)} \partial_t^{\alpha+1} u = \Delta u. \tag{8}$$

Subsequently, Chen and Holm extended their own work by including the fractional Laplacian operator in space [6]. Kelly et al. [7] also modified the aforementioned wave equation by adding a second time-fractional term to arrive at the power law wave equation. Recently, starting from the characteristic impedance and propagation coefficient, Chen et al. [8]

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