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Existence and multiplicity results of homoclinic solutions for fractional Hamiltonian systems[☆]

Yong Zhou[a,b,*], Lu Zhang[a]

[a] Faculty of Mathematics and Computational Science, Xiangtan University, Hunan 411105, PR China

[b] Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia

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ABSTRACT

In this paper, by the critical point theory, we consider the existence and multiplicity of solutions for the following fractional differential equation

$$ {}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_t u(t)) + L(t)u(t) = \nabla W(t, u(t)), \quad t \in \mathbb{R}, $$

where $\alpha \in (\frac{1}{2}, 1]$, ${}_{-\infty}D^{\alpha}_t$ and ${}_tD^{\alpha}_{\infty}$ are left and right Liouville–Weyl fractional derivatives of order α on the whole axis $\mathbb{R}$ respectively, $u \in \mathbb{R}^n$, $L(t)$ is positive definite symmetric matrix for all $t \in \mathbb{R}$ and $W : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a suitably chosen function.

1. Introduction and motivation

Hamiltonian systems is a significant field of nonlinear functional analysis, since they arise in phenomena studied in several fields of applied science such as physics, astronomy, chemistry, biology, engineering and other fields of science. Since Newton wrote the differential equation describing the motion of the planet and derived the Kepler ellipse as its solution, the complex dynamical behavior of the Hamiltonian system has attracted a wide range of mathematicians and physicists. The variational methods to investigate Hamiltonian system were first used by Poincare, who used the minimal action principle of the Jacobi form to study the closed orbits of a conservative system with two degrees of freedom. Ambrosetti and Rabinowitz in [1] proved "Mountain Pass Theorem", "Saddle point theorem", "Linking theorem" and a series of very important minimax form of critical point theorem. The study of Hamiltonian systems makes a significant breakthrough, due to critical point theory. Critical point theorem was first used by Rabinowitz [2] to obtain the existence of periodic solutions for first order Hamiltonian systems, while the first multiplicity result is due to Ambrosetti and Zelati [3]. Since then, there is a large number of literatures on the use of critical point theory and variational methods to prove the existence of homoclinic or heteroclinic orbits of Hamiltonian systems (see [4–12] and the references therein).

On the other hand, fractional calculus has received increased popularity and importance in the past decades to describe long-memory processes. For more details, we refer the reader to the monographs [13–16] and the articles [17–19], and the reference therein. Recently, the critical point theory has become an effective tool in studying the existence of solutions to fractional differential equations by constructing fractional variational structures. In [20], Jiao and Zhou were the first who

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* Corresponding author at: Faculty of Mathematics and Computational Science, Xiangtan University, Hunan 411105, PR China.
E-mail address: yzhou@xtu.edu.cn (Y. Zhou).

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use critical point theory to study the existence of solutions to the fractional boundary value problem

$$\begin{cases} {}_tD_T^\alpha({}_0D_t^\alpha(t)) = \nabla W(t,u(t)), \\ u(0) = u(T). \end{cases}$$

The authors obtained the existence of at least one nontrivial solution by using critical point theory. Next, Jiao and Zhou [21] considered a class of fractional boundary value problems

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \dfrac{1}{2}{}_tD_T^{-\beta}(u'(t))\right) + F(t,u(t)) = 0, \quad \text{a.e.} t \in [0,T], \\ u(0) = u(T) = 0. \end{cases}$$

They established the variational structure and obtained various criteria on the existence of solutions. Motivated by the above work, more and more authors began considering fractional Hamiltonian systems, see [22–28]. For example, in [26], the author considered the following fractional Hamiltonian systems:

$$\begin{cases} {}_tD_\infty^\alpha({}_{-\infty}D_t^\alpha u(t)) + L(t)u(t) = \nabla W(t,u(t)), \\ u \in H^\alpha(\mathbb{R}) \end{cases} \tag{FHS}$$

where $\alpha \in (\frac{1}{2}, 1]$, $t \in \mathbb{R}$, $u \in \mathbb{R}^n$, and $L \in C(\mathbb{R}, \mathbb{R}^{n^2})$ are symmetric and positive definite matrices for all $t \in \mathbb{R}$, and $W \in C^1(\mathbb{R}\times\mathbb{R}^n, \mathbb{R})$, and $\nabla W(t,u)$ is the gradient of W at u. The author showed that (FHS) possesses at least one nontrivial solution via Mountain Pass Theorem, by assuming that L and W satisfy the following hypotheses:

(L) L is a positive definite symmetric matrix for all $t \in \mathbb{R}$, and there exists a function $l(t) \in C(\mathbb{R}, (0,\infty))$ such that $l(t) \to \infty$ as $|t| \to \infty$ and

$$(L(t)x, x) \geq l(t)|x|^2, \quad \text{for all } t \in \mathbb{R},\ x \in \mathbb{R}^n.$$

(W_1) $W \in C^1(\mathbb{R}\times\mathbb{R}^n, \mathbb{R})$ and there is a constant $\mu > 2$ such that

$$0 < \mu W(t,x) \leq (x, \nabla W(t,x)), \quad \text{for all } t \in \mathbb{R},\ x \in \mathbb{R}^n \setminus \{0\}.$$

(W_2) $|\nabla W(t,x)| = o(|x|)$ as $x \to 0$ uniformly with respect to $t \in \mathbb{R}$.

(W_3) there exists $\overline{W} \in C(\mathbb{R}^n, \mathbb{R})$ such that

$$|W(t,x)| + |\nabla W(t,x)| \leq |\overline{W}(x)|, \quad \text{for every } x \in \mathbb{R}^n \text{ and } t \in \mathbb{R}.$$

In this paper, we study the existence and multiplicity of solutions for problem (FHS) by assuming the conditions weaker than the condition (L). Precisely, we suppose that

(L_1) $L \in C(\mathbb{R}, \mathbb{R}^{n\times n})$ is a positive definite symmetric matrix for all $t \in \mathbb{R}$ and there exists a function $l(t) \in C(\mathbb{R}, \mathbb{R}^+)$ such that $\inf_{t\in\mathbb{R}} l(t) > 0$ and

$$(L(t)u, u) \geq l(t)|u|^2, \quad \text{for all } t \in \mathbb{R},\ x \in \mathbb{R}^n;$$

(L_2) for any $b > 0$, there exists $T_b > 0$ such that

$$mes(\Lambda^b) = 0,$$

where $\Lambda^b := \{|t| > T_b :\ l(t) \ngeq b\}$.

Then, we give the first main result.

Theorem 1.1. *Assume that* (L_1), (L_2) *and* (W_1)–(W_3) *hold. Then system (FHS) possesses at least one nontrivial homoclinic solution. Moreover, if*

(W_4) $W(t,-u) = W(t,u)$, *for all* $(t,u) \in \mathbb{R}\times\mathbb{R}^n$,

holds, then (FHS) has infinitely many nontrivial homoclinic solutions.

Next, we consider the case that L is uniformly bounded from below, and does not necessarily satisfy the coercive condition. In precise terms, we need the following assumptions:

(L_3) $L \in C(\mathbb{R}, \mathbb{R}^{n\times n})$ is a positive definite symmetric matrix for all $t \in \mathbb{R}$ and there exists a constant $m > 0$ such that

$$(L(t)u, u) \geq m|u|^2, \quad \text{for all } t \in \mathbb{R},\ u \in \mathbb{R}^n.$$

In order to establish our main results, we assume that $W(t,u) = a(t)V(u)$ such that the following conditions hold:

(W_5) $a : \mathbb{R} \to \mathbb{R}^+$ is a continuous function such that $a(t) \to 0$ as $|t| \to \infty$;

(W_6) $V \in C^1(\mathbb{R}\times\mathbb{R}^n, \mathbb{R})$ and there is a constant $\mu > 2$ such that

$$0 < \mu V(u) \leq (\nabla V(u), u), \quad \text{for all } u \in \mathbb{R}^N \setminus \{0\};$$