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Applications of fractional calculus in solving Abel-type integral equations: Surface–volume reaction problem

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ABSTRACT

In this paper we consider a class of partial integro-differential equations of fractional order, motivated by an equation which arises as a result of modeling surface–volume reactions in optical biosensors. We solve these equations by employing techniques from fractional calculus; several examples are discussed. Furthermore, for the first time, we encounter an order of the fractional derivative other than $\frac{1}{2}$ in an applied problem. Hence, in this paper we explore the applicability of fractional calculus in real-world applications, further strengthening the true nature of fractional calculus.

1. Introduction

Until very recently, the fractional calculus had been a purely mathematical tool without apparent applications. Currently, fractional dynamical equations play a major role in modeling of anomalous behavior and memory effects, which are common characteristics of natural phenomena [1–3]. The fact that fractional derivatives introduce a convolution integral with a power-law memory kernel makes the fractional differential equations an important model to describe memory effects in complex systems. Thus, it is seen that fractional derivatives or integrals appear naturally when modeling long-term behaviors, especially in the areas of viscoelastic materials and viscous fluid dynamics [4,5].

Abel's study of the tautochrone problem [6] is considered to be the first application of fractional calculus to an engineering problem. In it one finds the path where the time it takes for an object to fall under the influence of gravity is independent of the initial position. The solution, which was solved using a fractional calculus approach, is now known to be a part of the inverted cycloid [6,7].

Now it is not hard to find very interesting and novel applications of fractional differential equations in physics, chemistry, biology, engineering, finance and other areas of sciences that have been developed in the last few decades. Some of the applications include: diffusion processes [8,9], mechanics of materials [10,11], combinatorics [12,13], inequalities [14], analysis [15,16], calculus of variations [17–22], signal processing [23], image processing [24], advection and dispersion of solutes in porous or fractured media [25], modeling of viscoelastic materials under external forces [26], bioengineering [27], relaxation and reaction kinetics of polymers [28], random walks [29], mathematical finance [30], modeling of combustion [31], control theory [32], heat propagation [33], modeling of viscoelastic materials [34] and even in areas such as psychology [35,36]. The list is by no means complete. It is easy to find hundreds, if not thousands, of new applications in which the fractional calculus approach is more than welcome.

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This is the first in a series of papers that seeks to find further potential applications of fractional calculus in solving real-world problems, a journey that can benefit both the understanding of profound complexities in the application, and the field of fractional calculus itself. As an application of the theory developed in this paper, we consider the surface-volume reaction problem. The governing equations of the mathematical formulation of such models naturally give rise to a nonlinear equation that contains a fractional integral embedded in it, and which has no solutions to date. Thus, in this paper we both extend the theory of fractional calculus methods by considering equations motivated by modeling the surface-volume reactions, and explore another interpretation of the fractional integral.

The remainder of this paper is organized as follows. The definitions and basic results are given in Section 2. In Section 3, we give the main results, which are generalizations of Abel's integral approach to the tautochrone problem. In Section 4, we give illustrative examples to motivate our approaches. One of the main examples is the surface-volume reaction problem that has several very interesting applications in mathematical biology and engineering [37].

2. Basic definitions and preliminary results

We adopt definitions given in [38] or in the encyclopaedic book by Samko et al. [39] here. We begin by introducing the concept of a *Riemann–Liouville fractional integral*:

Definition 2.1 (*[38]*)**.** Let $\alpha > 0$ with $n-1 < \alpha \leq n,\ n \in \mathbb{N}$, and $a < x < b$. The left- and right-Riemann–Liouville fractional integrals of order α of a function f are given by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\,dt \quad \text{and} \quad J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\,dt$$

respectively, where $\Gamma(\cdot)$ is Euler's gamma function defined by

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}\,dt.$$

Non-local fractional derivatives are defined via fractional integrals [39–41], while the local fractional derivatives are defined via a limit-based approach [42,43]. A new class of controlled-derivative approach appeared in [44]. A criteria to test whether a given derivative is a fractional derivative appeared in [45,46]. Among other approaches, in this work, we utilize only the non-local Volterra-type definitions for the fractional derivative given below.

Definition 2.2 (*[38]*)**.** The left- and right-*Riemann–Liouville fractional derivatives* of order $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}$, are defined by

$$D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x (x-t)^{n-\alpha-1}f(t)\,dt,$$
$$D^{\alpha}_{b-}f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_x^b (t-x)^{n-\alpha-1}f(t)\,dt,$$

respectively. It can be shown that in the case of $\alpha \in \mathbb{N}$ the above definitions coincide with the standard definition of the nth-derivative of $f(x)$.

Definition 2.3 (*[38]*)**.** The left- and right-*Caputo fractional derivatives* of order $\alpha > 0$, $n-1 < \alpha < n,\ n \in \mathbb{N}$, are defined by

$${}^{C}D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1}f^{(n)}(t)\,dt \quad \text{and} \quad {}^{C}D^{\alpha}_{b-}f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b (t-x)^{n-\alpha-1}f^{(n)}(t)\,dt,$$

respectively. It can be shown that in the case of $\alpha \in \mathbb{N}$ the above definitions reduce to the standard definition of the nth-derivative of $f(x)$. To see this, let us assume that $0 \leq n-1 < \alpha < n$, and $f(x) \in C^{n+1}[a,T]$. Then in the case of Caputo's derivative, we have, by integration by parts [38, p. 79],

$$\lim_{\alpha\to n} {}^{C}D^{\alpha}_{a+}f(x) = \lim_{\alpha\to n}\left[\frac{f^{(n)}(a)(x-a)^{n-a}}{\Gamma(n-\alpha+1)} + \frac{1}{\Gamma(n-\alpha+1)}\int_a^x (x-\tau)^{n-\alpha}f^{(n+1)}(\tau)\,d\tau\right] \tag{1}$$
$$= f^{(n)}(a) + \int_a^x f^{(n+1)}(\tau)\,d\tau = f^{(n)}(x), \quad n = 1, 2, \ldots. \tag{2}$$

This shows that the Caputo derivative is a generalization of the integer-order derivative. A different proof of the fact in question, which does not use integration by parts, can be found in [47, pp. 49, 51] using an equivalent definition of the

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