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Unidirectional flows of fractional Jeffreys' fluids: Thermodynamic constraints and subordination

Emilia Bazhlekova *, Ivan Bazhlekov

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev Str., Bl. 8, Sofia 1113, Bulgaria

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ABSTRACT

A class of initial–boundary value problems governing the velocity distribution of unidirectional flows of viscoelastic fluids is studied. The generalized fractional Jeffreys' constitutive model is used to describe the viscoelastic properties. Thermodynamic constraints on the parameters of the model are derived from the monotonicity of the corresponding relaxation function. Based on these constraints, a subordination principle for the considered class of problems is established. It gives an integral representation of the solution in terms of a probability density function and the solution of a related wave equation. Explicit representation of the probability density function is derived from the solution of the Stokes' first problem. Numerical verification of the obtained analytical results is provided.

1. Introduction

Increasing attention has been devoted to the prediction of behavior of viscoelastic non-Newtonian fluids in recent years, due to their various applications (molten plastics, oils and greases, suspensions, emulsions, etc.) The Jeffreys' fluid is a non-Newtonian rate-type fluid, which constitutive equation has the dimensionless form

$$\left(1 + a\frac{\partial}{\partial t}\right)\sigma = \left(1 + b\frac{\partial}{\partial t}\right)\dot{\varepsilon}, \qquad (1)$$

where σ is the shear stress, $\dot{\varepsilon}$ is the rate of strain, a and b are the dimensionless relaxation and retardation times, respectively, with $a > b > 0$. This model has become very popular since it can describe many of the non-Newtonian characteristics of polymeric suspensions [1,2].

Fractional calculus has been extensively used in linear viscoelasticity [3–7]. Due to the non-locality of fractional derivatives, fractional order models provide a higher level of adequacy, preserving linearity, and give the possibility for relatively simple description of the complex behavior of non-Newtonian viscoelastic fluids. The fractional Jeffreys' model introduced in [8] (also referred to as fractional Oldroyd-B model) is derived from (1) by replacing the terms $a\frac{\partial}{\partial t}$ and $b\frac{\partial}{\partial t}$ by fractional differential operators aD_t^{α} and bD_t^{β}, respectively, with $0 < \alpha, \beta < 1$. In [8] a good fit with experimental data for this model is achieved.

Unidirectional flows of fractional Jeffreys' fluids in different geometries and under various conditions (e.g. porous media, magneto-hydrodynamic effects, slip conditions) are studied in [9–14], to mention only few of many recent publications. In these works solutions of the initial–boundary value problems (IBVPs) for the velocity distribution are obtained in the form

* Corresponding author.
E-mail addresses: e.bazhlekova@math.bas.bg (E. Bazhlekova), i.bazhlekov@math.bas.bg (I. Bazhlekov).

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of series expansions. The considered IBVPs for unidirectional flows of fractional Jeffreys' fluids usually can be written in abstract form as

$$\left(1 + aD_t^\alpha\right) u_t = (1 + bD_t^\beta)Au + \left(1 + aD_t^\alpha\right) f(t), \quad t > 0, \tag{2}$$

$$u(0) = u_t(0) = 0, \tag{3}$$

where A is a linear operator defined in a suitably chosen Banach space X (A is usually a one- or two-dimensional realization of the Laplace operator, or a more general elliptic operator) and f is an X-valued function.

Regarding the orders α and β of the fractional derivatives different assumptions are considered in the literature: e.g. in [8] the restriction $\alpha \geq \beta$ is imposed, in [12,13] it is assumed that $\alpha \leq \beta$, while in other works the whole range $0 < \alpha, \beta < 1$ is considered. On the other hand, applying the technique developed by Bagley and Torvik in [4], it is proven in [15] that the fractional Jeffreys' model is consistent with the second law of thermodynamics if and only if

$$\alpha = \beta \quad \text{and} \quad a \geq b. \tag{4}$$

However, to the best of our knowledge, the impact of thermodynamic restrictions (4) on the solution of the corresponding problem (2)–(3) has not been analyzed.

Stokes' first problem is concerned with a specific case of a unidirectional flow set into motion by a sudden movement of a flat plate. One of the first studies on Stokes' first problem for classical Jeffreys' fluids is [16]. Since then many works have been devoted to this problem and its fractional order generalizations. As pointed out in [17,18], in a number of recent works the obtained exact solutions of Stokes' first problem for classical Jeffreys' fluids contain mistakes, which propagate as well to their fractional order generalizations.

Motivated by the aforementioned developments, in this work we first revisit the fractional order Jeffreys' constitutive model and the Stokes' first problem for thermodynamically compatible fractional Jeffreys' fluids. Thermodynamic constraints (4) are derived from the monotonicity properties of the corresponding relaxation function and exact solutions for the Stokes' first problems in half-space and on a strip are obtained. Next, the general problem (2)–(3) is studied, where the operator A is a generator of a cosine family (for definition see e.g. [19], Section 3.14). Our main result is the following subordination representation for the solution operator $S(t)$ of this problem in the case when thermodynamic constraints (4) are satisfied:

$$S(t) = \int_0^\infty \varphi(t,\tau)T(\tau)\,\mathrm{d}\tau, \quad t > 0, \tag{5}$$

where $\varphi(t,\tau)$ is a probability density function (p.d.f.) in τ, and $T(t)$ is the cosine family generated by the operator A. By the variation of parameters formula the solution of (2)–(3) is then given by

$$u(t) = \int_0^t S(t-\sigma)f(\sigma)\,\mathrm{d}\sigma. \tag{6}$$

The p.d.f. $\varphi(t,\tau)$ can be expressed via the solution of the Stokes' first problem in half-space. This provides an explicit integral representation for $\varphi(t,\tau)$.

Subordination principle in a general setting is introduced in [20], Chapter 4. Various applications of this principle have been found so far: in inverse problems [21], asymptotic analysis of fractional diffusion-wave equations [22], stochastic solutions [23], abstract Volterra integro-differential equations [24], operator theory [25], etc.

This paper is organized as follows. In Section 2 the relaxation function of the fractional Jeffreys' model is studied and the thermodynamic constraints (4) are derived from its monotonicity. In Section 3 explicit solutions for Stokes' first problems in half-space and on a strip are derived and their properties are discussed. Section 4 is devoted to the derivation of the subordination formula and its applications. Numerical results based on the obtained analytical representations are given in Section 5 and independent numerical checks are performed. Some facts concerning completely monotone functions and related classes of functions are summarized in an Appendix.

2. Thermodynamic restrictions and relaxation function

Consider a unidirectional viscoelastic flow and suppose it is quiescent for all times prior to some starting time that we assume as $t = 0$. Since we work only with causal functions ($f(t) = 0$ for $t < 0$) if there is no danger of confusion for the sake of brevity we still denote by $f(t)$ the function $H(t)f(t)$, where $H(t)$ is the Heaviside unit step function.

According to the generalized fractional Jeffreys' model the relationship between stress $\sigma(t)$ and strain $\varepsilon(t)$ is given by the following linear constitutive equation [8,15]

$$\left(1 + aD_t^\alpha\right)\sigma(t) = (1 + bD_t^\beta)\dot{\varepsilon}(t) \tag{7}$$

where $a, b > 0$, $0 < \alpha, \beta \leq 1$, the over-dot denotes the first time derivative, and D_t^α, D_t^β are Riemann–Liouville fractional time derivatives:

$$D_t^\gamma f(t) = \frac{1}{\Gamma(1-\gamma)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{f(\tau)}{(t-\tau)^\gamma}\,\mathrm{d}\tau, \quad \gamma \in (0,1),$$

where $\Gamma(\cdot)$ is the Gamma function, $D_t^1 = \mathrm{d}/\mathrm{d}t$.