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A mixture of thermoelastic solids with two temperatures

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ABSTRACT

In this work we study, from the numerical point of view, a problem involving one-dimensional thermoelastic mixtures with two different temperatures; that is, when each component of the mixture has its own temperature. The mechanical problem consists of two hyperbolic equations coupled with two parabolic equations. The variational problem is derived in terms of product variables. An existence and uniqueness result and an energy decay property are stated. Then, fully discrete approximations are introduced using the finite element method and the backward Euler scheme. A discrete stability property is proved and a priori error estimates are obtained, from which the linear convergence is deduced. Finally, some numerical simulations are described to show the accuracy of the approximation and the behavior of the solution.

1. Introduction

The theory of interacting continua has received an increasing interest during the last years because the application of this kind of materials in the industry has grown drastically as, for instance, the use of composites in the automotive industry. Another application has been to model swelling soils by means of a three components mixture [1]. Since the first mathematical works published by Truesdell and Toupin [2] or Green and Naghdi [3,4], a large number of papers involving mixtures of elastic and viscoelastic constituents have considered different analysis such as existence, uniqueness, continuous dependence or asymptotic stability (see, for instance, [5–18] or the book [19]).

Eringen and Ingram [20,21] presented the first thermomechanical theory for mixtures of Newtonian fluids in which the two constituents may have different temperatures; that is, each constituent of the mixture was assumed to have its temperature. Other constitutive theories were also proposed (see, e.g., [22–27]). Ieşan derived in [28] the basic equations of a nonlinear theory for binary mixtures of thermoelastic materials. Moreover, the linear theory was also shown. This problem was considered later in [29], restricted to the one-dimensional case. In this work, we revisit the problem studied in [29], where the existence and uniqueness of solutions to a problem involving thermoelastic mixtures were considered. The energy decay of the solutions was also analyzed depending on the coupling between the displacements and the temperatures. Here, we continue this paper, introducing a finite element approximation of the problem, providing some a priori error estimates and performing some numerical simulations.

The paper is organized as follows. In Section 2, following [28,29] the model and its variational formulation are presented in terms of product variables. An existence and uniqueness result and an energy decay result are stated. Then, in Section 3 a numerical approximation based on the finite element method and the backward Euler scheme is proposed and analyzed. Finally, numerical simulations are described in Section 4.

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2. The model and its variational formulation

In this section, following [28,29] we describe the model, the variational formulation of the mechanical problem and the required assumptions, and we state the existence of a unique solution and an energy decay property. We refer the reader to [29] for details.

Let $[0,\ell]$, $\ell>0$, be the one-dimensional rod of length ℓ and denote by $[0,T]$, $T>0$, the time interval of interest. Moreover, let $x\in[0,\ell]$ and $t\in[0,T]$ be the spatial and time variables, respectively. In order to simplify the writing, we do not indicate the dependence of the functions on x and t, and a subscript under a variable represents its derivative with respect to the prescribed variable.

According to [29], we assume that the rod is composed of a mixture of two interacting continua occupying the interval $[0,\ell]$, where the respective displacements are denoted by u and w. Moreover, we assume the existence of two different temperatures, in each point x and at time t, given by θ_1 and θ_2.

We assume that the rod it fixed at its corners $x=0,\ \ell$ and so, $u(0,t)=u(\ell,t)=w(0,t)=w(\ell,t)=0$ for all $t\in[0,T]$. Furthermore, in order to simplify the calculations we also assume that the two temperatures vanish at both ends and so $\theta_1(0,t)=\theta_1(\ell,t)=\theta_2(0,t)=\theta_2(\ell,t)=0$ for all $t\in[0,T]$. We note that it is easy to extend the results presented in this paper to more general situations.

Therefore, the mechanical problem of a mixture of thermoelastic solids with two different temperatures is written as follows (see [29]).

Problem P. Find the displacement of the first constituent $u:[0,\ell]\times[0,T]\to\mathbb{R}$, the displacement of the second constituent $w:[0,\ell]\times[0,T]\to\mathbb{R}$, the temperature of the first constituent $\theta_1:[0,\ell]\times[0,T]\to\mathbb{R}$ and the temperature of the second constituent $\theta_2:[0,\ell]\times[0,T]\to\mathbb{R}$ such that

$$\rho_1 u_{tt}-a_{11}u_{xx}-a_{12}w_{xx}+\alpha(u-w)+\beta_1\theta_{1,x}+\beta_2\theta_{2,x}=0 \quad \text{in } (0,\ell)\times(0,T), \tag{1}$$

$$\rho_2 w_{tt}-a_{12}u_{xx}-a_{22}w_{xx}-\alpha(u-w)+\gamma_1\theta_{1,x}+\gamma_2\theta_{2,x}=0 \quad \text{in } (0,\ell)\times(0,T), \tag{2}$$

$$b_1\theta_{1,t}+b_2\theta_{2,t}-K_{11}\theta_{1,xx}-K_{12}\theta_{2,xx}+\beta_1u_{xt}+\beta_2w_{xt}+a(\theta_1-\theta_2)=0 \quad \text{in } (0,\ell)\times(0,T), \tag{3}$$

$$b_2\theta_{1,t}+b_3\theta_{2,t}-K_{21}\theta_{1,xx}-K_{22}\theta_{2,xx}+\gamma_1u_{xt}+\gamma_2w_{xt}-a(\theta_1-\theta_2)=0 \quad \text{in } (0,\ell)\times(0,T), \tag{4}$$

$$u(0,t)=w(0,t)=u(\ell,t)=w(\ell,t)=0 \quad \text{for a.e.} \quad t\in(0,T), \tag{5}$$

$$\theta_1(0,t)=\theta_1(\ell,t)=\theta_2(0,t)=\theta_2(\ell,t)=0 \quad \text{for a.e. } t\in(0,T), \tag{6}$$

$$u(x,0)=u_0(x), \qquad w(x,0)=w_0(x) \quad \text{for a.e. } x\in(0,\ell), \tag{7}$$

$$\theta_1(x,0)=\theta_{10}(x), \qquad \theta_2(x,0)=\theta_{20}(x) \quad \text{for a.e. } x\in(0,\ell), \tag{8}$$

$$u_t(x,0)=v_0(x), \qquad w_t(x,0)=e_0(x) \quad \text{for a.e. } x\in(0,\ell). \tag{9}$$

Here, ρ_1 and ρ_2 are the mass densities of each constituent, $(a_{ij})_{i,j=1}^2$ and α are elastic coefficients, b_1, b_2 and b_3 denote the heat capacity coefficients, β_1, β_2, γ_1 and γ_2 are thermal expansion coefficients, $(K_{ij})_{i,j=1}^2$ represent thermal diffusion coefficients, a is a heat coefficient and u_0, v_0, w_0, e_0, θ_{10} and θ_{20} are given initial conditions.

In order to obtain the variational formulation of Problem P, let $Y=L^2(0,\ell)$ and denote by $(\cdot,\cdot)$ the scalar product in this space, with corresponding norm $\|\cdot\|$. Moreover, let us define the variational space E as follows,

$$E=\{w\in H^1(0,\ell);\ w(0)=w(\ell)=0\},$$

with scalar product $(\cdot,\cdot)_E$ and norm $\|\cdot\|_E$.

By using the integration by parts and the Dirichlet boundary conditions at $x=0,\ \ell$, we write the variational formulation of Problem P in terms of the velocity $v=u_t$ of the first constituent, the velocity $e=w_t$ of the second constituent and the temperatures θ_1 and θ_2.

Find the velocity of the first constituent $v:[0,T]\to E$, the velocity of the second constituent $e:[0,T]\to E$, the temperature of the first constituent $\theta_1:[0,T]\to E$ and the temperature of the second constituent $\theta_2:[0,T]\to E$ such that $v(0)=v_0$, $e(0)=e_0$, $\theta_1(0)=\theta_{10}$, $\theta_2(0)=\theta_{20}$ and, for a.e. $t\in(0,T)$,

$$\rho_1(v_t(t),z)+a_{11}(u_x(t),z_x)+a_{12}(w_x(t),z_x)+\alpha(u(t)-w(t),z)+\beta_1(\theta_{1,x}(t),z)+\beta_2(\theta_{2,x}(t),z)=0 \quad \forall z\in E, \tag{10}$$

$$\rho_2(e_t(t),r)+a_{12}(u_x(t),r_x)+a_{22}(w_x(t),r_x)-\alpha(u(t)-w(t),r)+\gamma_1(\theta_{1,x}(t),r)+\gamma_2(\theta_{2,x}(t),r)=0 \quad \forall r\in E, \tag{11}$$

$$b_1(\theta_{1,t}(t),\psi)+b_2(\theta_{2,t}(t),\psi)+K_{11}(\theta_{1,x}(t),\psi_x)+K_{12}(\theta_{2,x}(t),\psi_x)+\beta_1(v_x(t),\psi)$$
$$+\beta_2(e_x(t),\psi)+a(\theta_1(t)-\theta_2(t),\psi)=0 \quad \forall\psi\in E, \tag{12}$$

$$b_2(\theta_{1,t}(t),\xi)+b_3(\theta_{2,t}(t),\xi)+K_{21}(\theta_{1,x}(t),\xi_x)+K_{22}(\theta_{2,x}(t),\xi_x)+\gamma_1(v_x(t),\xi)$$
$$+\gamma_2(e_x(t),\xi)-a(\theta_1(t)-\theta_2(t),\xi)=0 \quad \forall\xi\in E, \tag{13}$$

where the displacements of the first and second constituent are then recovered from the relations

$$u(t)=\int_0^t v(s)\,ds+u_0, \qquad w(t)=\int_0^t e(s)\,ds+w_0. \tag{14}$$

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