ARTICLE IN PRESS
Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Well-posedness for density-dependent Boussinesq equations without dissipation terms in Besov spaces

Hua Qiu [a,*], Zheng'an Yao [b]

[a] Department of Mathematics, South China Agricultural University, Guangzhou 510642, PR China
[b] School of Mathematics, Sun Yat-Sen University, Guangzhou 510275, PR China

ARTICLE INFO

Article history:
Received 13 September 2016
Received in revised form 8 February 2017
Accepted 27 February 2017
Available online xxxx

Keywords:
Boussinesq equations
Local well-posedness
Besov spaces
Density-dependent

ABSTRACT

In this paper, we consider the N-dimensional incompressible density-dependent Boussinesq equations without dissipation terms ($N \geq 2$). We establish the local well-posedness for the incompressible Boussinesq system under the framework of the Besov spaces. In addition, we also obtain a Beale–Kato–Majda-type regularity criterion.

1. Introduction

In this paper, we consider the following N-dimensional incompressible density-dependent Boussinesq equations without dissipation terms ($N \geq 2$):

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P = \rho\theta e_N + \rho f, \\ \partial_t \theta + u \cdot \nabla\theta = 0, \\ \nabla \cdot u = 0, \quad (x,t) \in \mathbb{R}^N \times (0,\infty), \\ (\rho, u, \theta)|_{t=0} = (\rho_0, u_0, \theta_0), \end{cases} \tag{1.1}$$

where $u(x,t) = (u_1(x,t), \ldots, u_N(x,t))$ denotes the fluid velocity vector field, $P = P(x,t)$ is the scalar pressure, $\theta = \theta(x,t)$ is the scalar temperature, $e_N = (0, \ldots, 0, 1)$, while ρ_0, u_0 and θ_0 are the given initial density, initial velocity and initial temperature respectively, with $\nabla \cdot u_0 = 0$.

The Boussinesq system describes the motion of lighter or denser incompressible fluid under the influence of gravitational forces, and has important roles in the atmospheric sciences [1], as well as a model in many geophysical applications [2]. For this reason, this system is studied systematically by scientists from different domains. The global-in-time regularity of solutions to (1.1) with dissipation terms and $\rho = C$ (C is constant) is well-known in the two-dimensional case [3]. However, the question about the regularity of singularity without dissipation terms is an open problem in mathematical fluid mechanics [4–8] as ρ is constant. Recently, the global regularity of solutions to the problem (1.1) with partial viscosity terms has been studied by many authors [9–15], as ρ is the constant. For the three-dimensional density-dependent Boussinesq

* Corresponding author.
E-mail addresses: tsiuhua@scau.edu.cn (H. Qiu), mcsyao@mail.sysu.edu.cn (Z. Yao).

http://dx.doi.org/10.1016/j.camwa.2017.02.041
0898-1221/

Please cite this article in press as: H. Qiu, Z. Yao, Well-posedness for density-dependent Boussinesq equations without dissipation terms in Besov spaces, Computers and Mathematics with Applications (2017), http://dx.doi.org/10.1016/j.camwa.2017.02.041

equations, Fan and Ozawa [16] proved some regularity criteria of strong solutions; Xu [17] obtained the local existence and a blow-up criterion for smooth solutions to the 2-D isentropic compressible Boussinesq equations by using some new commutator estimates. Further discussions can be found in [18–40] and the references therein.

The system (1.1) reduces to the incompressible density-dependent Euler equations, when $\theta = 0$. Chae [41] showed the local well-posedness of the incompressible density-dependent Euler equations in the critical Besov spaces, i.e., for $\rho_0, u_0 \in B^{\frac{N}{2}+1}_{2,1}$. Very recently, Zhou et al. [42] generalized the result of [41] under the conditions $\rho_0, u_0 \in B^{\frac{N}{p}+1}_{p,1}$ with $1 < p < \infty$.

The purpose of this paper is to establish the local well-posedness for the density-dependent Boussinesq system (1.1) and consider regularity condition of the smooth solutions for this system under the framework of the Besov spaces. In this paper, we will adapt the methods of Chae [41] and Zhou et al. [42] by means of the Littlewood–Paley decomposition and Bony's paradifferential calculus.

To simplify the presentation, in this paper, we assume that $0 < m < \rho_0(x) < M < +\infty$ and $\lim_{x\to\pm\infty} \rho_0 = \bar{\rho}$. Without loss of generality, we may assume that $\bar{\rho} = 1$, and let $\sigma = \rho - 1$. Precisely, from above, then we rewrite the above system (1.1) as:

$$\begin{cases} \partial_t \sigma + u \cdot \nabla \sigma = 0, \\ \partial_t u + u \cdot \nabla u + \dfrac{1}{\rho} \nabla P = \theta e_N + f, \\ \partial_t \theta + u \cdot \nabla \theta = 0, \\ \nabla \cdot u = 0, \\ \sigma = \rho - 1, \\ (\rho, u, \theta)|_{t=0} = (\rho_0, u_0, \theta_0). \end{cases} \tag{1.2}$$

Our first main result is stated as follows:

Theorem 1.1 (*Local Existence*). *Let $N \geq 2$, $1 < p < \infty$. Suppose that $\sigma_0 = \rho_0 - 1 \in B^{\frac{N}{p}+1}_{p,1}(\mathbb{R}^N)$, $u_0 \in B^{\frac{N}{p}+1}_{p,1}(\mathbb{R}^N)$, $\theta_0 \in B^{\frac{N}{p}+1}_{p,1}(\mathbb{R}^N)$, $f \in L^1_T(B^{\frac{N}{p}+1}_{p,1}(\mathbb{R}^N))$ with $\nabla \cdot u_0$. Then there exists a positive time $T = T(\|\rho_0\|_{B^{\frac{N}{p}+1}_{p,1}}, \|u_0\|_{B^{\frac{N}{p}+1}_{p,1}}, \|\theta_0\|_{B^{\frac{N}{p}+1}_{p,1}}) > 0$ such that the system (1.2) has a unique solution $(\rho, u, \theta) \in C([0, T]; B^{\frac{N}{p}+1}_{p,1}(\mathbb{R}^N))$.*

Furthermore, we get the following blow-up criterion:

Theorem 1.2 (*Blow-up Criterion*). *Let ρ_0, u_0, θ_0 and $f(t)$ be given as in the above. Then, the local solution $(\rho, u, \theta) \in C([0, T]; B^{\frac{N}{p}+1}_{p,1})$ constructed in Theorem 1.1 blows up at time T^* if and only if*

$$\int_0^{T^*} \|\nabla \times u(t)\|_{\dot{B}^{\frac{N}{p}}_{p,1}} dt = \infty.$$

This paper is structured as follows. In Section 2, we give the preliminaries and some basic facts. In Section 3, we provide the proof of Theorem 1.1. In Section 4, we give the proof of Theorem 1.2.

Notation Throughout this paper, the constant C stands for the generic constant. Let X be a Banach space. For $p \in [1, +\infty]$, the notation $L^p([0, T]; X)$ stands for the set of measurable functions on $[0, T]$ with values in X, such that $t \longmapsto \|f(t)\|_X$ belongs to $L^p([0, T])$. For simplicity, we denote $L^p([0, T]; X)$ by $L^p_T(X)$. Furthermore, $\|\cdot\|_p$ denotes the norm of the Lebesgue space $L^p(X)$, and $\int_{\mathbb{R}^N} dx$ by $\int dx$.

2. Preliminaries

In this section, we give some lemmas and introduce some basic facts on Littlewood–Paley theory, which will be used in the proof of our main results.

Let $\mathcal{S}(\mathbb{R}^N)$ be the Schwarz class of rapidly decreasing functions. Choose $\varphi \in \mathcal{S}(\mathbb{R}^N)$ supported in $\mathcal{C} \triangleq \{\xi \in \mathbb{R}^N, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ such that

$$\sum_{q\in\mathbb{Z}} \varphi(2^{-q}\xi) = 1, \quad \xi \in \mathbb{R}^N - \{0\}.$$

Denote $h = \mathcal{F}^{-1}\varphi$. The dyadic blocks are defined as follows respectively.

$$\Delta_q u \triangleq \varphi(2^{-q}D)u = 2^{qN} \int_{\mathbb{R}^N} h(2^q y) u(x - y) dy,$$

$$S_q u \triangleq \sum_{k\leq q-1} \Delta_q u = \mathcal{X}(2^{-q}D)u$$

Please cite this article in press as: H. Qiu, Z. Yao, Well-posedness for density-dependent Boussinesq equations without dissipation terms in Besov spaces, Computers and Mathematics with Applications (2017), http://dx.doi.org/10.1016/j.camwa.2017.02.041