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A fast preconditioned policy iteration method for solving the tempered fractional HJB equation governing American options valuation

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ABSTRACT

A fast preconditioned policy iteration method is proposed for the Hamilton–Jacobi–Bellman (HJB) equation involving tempered fractional order partial derivatives, governing the valuation of American options whose underlying asset follows exponential Lévy processes. An unconditionally stable upwind finite difference scheme with shifted Grünwald approximation is first developed to discretize the established HJB equation under the tempered fractional diffusion models. Next, the policy iteration method as an outer iterative method is utilized to solve the discretized HJB equation and proven to be convergent in finite steps to its numerical solution. Given the Toeplitz-like structure of the coefficient matrix in each policy iteration, the resulting linear system can be fast solved by the Krylov subspace method as an inner iterative method via fast Fourier transform (FFT). Furthermore, a novel preconditioner is proposed to speed up the convergence rate of the inner Krylov subspace iteration with theoretical analysis to ensure the linear system can be solved in $\mathcal{O}(N \log N)$ operations under some mild conditions, where N is the number of spatial node points. Numerical examples are given to demonstrate the accuracy and efficiency of the proposed fast preconditioned policy method.

1. Introduction

An American option is a financial instrument that gives its buyer the right, but not the obligation, to buy (or sell) an asset at a predetermined price at any time step up to a certain terminal time T. This additional early exercise right, compared with a European option, casts the American option pricing problem into the following highly nonlinear and comparably intriguing linear complementarity problem (LCP) [1],

$$\begin{cases} \mathcal{L}_B V(x,t) \geq 0, \\ V(x,t) \geq V^*(x), \\ \mathcal{L}_B V(x,t) \cdot (V(x,t) - V^*(x)) = 0, \end{cases} \tag{1.1}$$

for $(x, t) \in \mathbb{R} \times [0, T)$ with the terminal condition $V(x, T) = V^*(x)$ where $V^*(x)$ denotes the payoff function of the option (e.g., $\max\{K - e^x, 0\}$ for a put option), $\mathcal{L}_B$ is a linear differential operator varied with the model assumption for the underlying price S_t, and $x = \ln S$.

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Black and Scholes (BS) [2] initiated the option pricing theory in 1973, under which the underlying price is assumed to follow a geometric Brownian motion with constant drift and volatility. However, it has been known for many years that the classic BS model suffers from some shortcomings and for example is not capable of explaining many important empirical facts of financial markets, like skewed and heavy tailed return distributions or large, sudden movements in stock prices. Thus, despite the superior analytical tractability of the geometric Brownian motion model, many authors proposed the more general class of exponential Lévy processes as the underlying model for prices of financial quantities [3], including the jump–diffusion [4,5] and infinite activity Lévy processes (finite moment log stable (FMLS) model [6], Carr–Geman–Madan–Yor (CGMY) model [7], KoBoL model [8], etc.).

The American option pricing problem under the framework of exponential Lévy processes is a widely discussed problem. According to [3], when the Brownian motion component is replaced by a Lévy process, the BS equation becomes a partial integro-differential equation (PIDE). Since most of such PIDEs are hardly solved in closed form, efficient numerical methods become essential. Six efficient methods based on a linear complementarity formulation and finite difference discretizations was given by Salmi and Toivanen [9] under finite activity jump–diffusion models. In this paper, we focus on the infinite activity Lévy processes. Almendral and Oosterlee in [10] and Wang, Wan and Forsyth in [11] considered pricing American options numerically under the CGMY process with different method to discretize the integral part. Recently, Cartea and del-Castillo-Negrete [12] creatively showed that for some particular Lévy processes, including CGMY and KoBoL models, the European barrier option value satisfies a tempered fractional partial differential equation (TFPDE). The application of fractional calculus to option pricing problem is increasingly recognized, due to the non-local nature of the fractional operator, which weights information of the option over a range of underlying values rather than narrowly focusing on some localized information. American option pricing problem under the FMLS model via fractional partial differential equation (FPDE) framework was initiated in [13] by using the power penalty method and extended in [14] by means of a predictor–corrector approach. American option pricing problem under the KoBoL model via TFPDE framework was considered in [15] by using the standard penalty method [16,17]. Currently, policy iteration method, developed in [18] for the numerical solution of Hamilton–Jacobi–Bellman (HJB) equations, was applied as an easy way of pricing American options [19]. Policy iteration method is based on the interpretation of LCP problem (1.1) as the following HJB equation,

$$\min\{\mathcal{L}_B V(x,t), V(x,t) - V^*(x)\} = 0. \tag{1.2}$$

For solving the above HJB equation numerically, the finite difference method with shifted Grünwald approximation proposed in [20] is used to discretize it. Next, the policy iteration method developed in [18] can be utilized for solving the discretized HJB equation and consequently a linear system needs to be solved per policy iteration. To the best of our knowledge, there have not been any papers on solving the discrete HJB equation with tempered fractional derivatives so far, and therefore this is one of main aims in this paper. To fastly solve the resultant linear system under tempered fractional diffusion models, there are several fast algorithms that have been proposed, such as fast conjugate gradient method for pricing double barrier options [21], preconditioned technique for pricing barrier options [22], and band preconditioner for pricing European options [23]. Given the Toeplitz-like matrix structure, which is distinct from the above-mentioned papers, a novel preconditioner based on approximating the inverse of coefficient matrix is proposed in this paper.

The contributions of the present paper are as follows. First of all, under the tempered fractional diffusion models, including the CGMY model and the KoBoL model, an unconditionally stable upwind finite difference scheme with shifted Grünwald approximation is used to discretize the HJB equation governing the American option price, where the tempered fractional derivative is defined in Grünwald–Letnikov sense [24], which gives rise to the M-Matrix structure of the coefficient matrix. Secondly, policy iteration method as an outer iteration is utilized as an easy and efficient way to solve the discretized HJB equation. Compared with the penalty method, it averts the question about how large is appropriate for the penalty parameter λ in [15]. Thirdly, the Toeplitz-like structure of the coefficient matrix is found per outer iteration and consequently the resulting linear system can be fastly solved by the Krylov subspace method as an inner iterative method via fast Fourier transform (FFT). To be noted, for such Toeplitz-like matrix, a novel preconditioner is implemented to accelerate convergence rate of this inner iteration. Last but not least, to show the effectiveness and efficiency of the proposed numerical method, it is applied to price the American put option and calculate its Greeks.

The structure of this paper is organized as follows. In Section 2, we introduce the Lévy processes and give the HJB equations involving tempered fractional derivatives for pricing American options under the specific models. In Section 3, an upwind finite difference scheme with shifted Grünwald approximation is constructed for the above-given HJB equations, and proven to be unconditionally stable. In Section 4, we introduce the fast policy iteration method to solve the discretized HJB equations with convergent analysis. In Section 5, a novel preconditioner is proposed to speed up the convergence rate of the inner Krylov subspace method. In Section 6, we validate our proposed method with numerical experiments. Conclusion will be given in Section 7.

2. Exponential Lévy processes

From [3], option pricing for pure jump processes will result in an incomplete market, which implies the risk-neutral measure is not unique. To choose the risk-neutral measures, see [3] for details. For this paper, we assume the risk-neutral measure $\mathbb{P}$ is given. We consider a more general financial framework: under the predefined filtered probability space