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Regularity criteria for some simplified non-isothermal models for nematic liquid crystals

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This paper is dedicated to Professor Gen Nakamura and Professor Fahuai Yi on the occasion of their 70th birthday

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ABSTRACT

This paper proves some regularity criteria for some simplified non-isothermal models for nematic liquid crystals.

1. Introduction

In this paper we first consider the following simplified nonisothermal model for nematic liquid crystals [1]:

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$\operatorname{div} u = 0, \tag{1.2}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla\pi - \operatorname{div}(\mu(\theta)(\nabla u + \nabla u^t)) = -\operatorname{div}(\nabla d \odot \nabla d), \tag{1.3}$$

$$\partial_t(\rho\theta) + \operatorname{div}(\rho\theta u) - \Delta\theta = 0, \tag{1.4}$$

$$\partial_t d + u \cdot \nabla d - \Delta d = d(1 - |d|^2), \tag{1.5}$$

$$(\rho, u, \theta, d)(\cdot, 0) = (\rho_0, u_0, \theta_0, d_0) \quad \text{in } \mathbb{R}^3. \tag{1.6}$$

Here the unknowns ρ, u, π, θ, and d denote the density, velocity, pressure, temperature, and the macroscopic orientations, respectively. $\mu(\theta)$ is the viscosity, we will assume that μ is a smooth function and satisfies

$$0 < \frac{1}{C_1} \le \mu(\theta) \le C_1 < \infty \quad \text{when } |\theta| \le C_2 < \infty \tag{1.7}$$

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for some positive constants $C_1 > 1$ and $C_2 > 0$. The symbol $\nabla d \odot \nabla d$ denotes a matrix whose (i, j)th entry is $\partial_i d \partial_j d$. u^t is the transpose of u and $\partial_t u \equiv u_t$.

Next, we consider the following system: (1.1)–(1.4), (1.6) with

$$\partial_t d + u \cdot \nabla d - \Delta d = |\nabla d|^2 d, \quad |d| = 1. \tag{1.8}$$

When $\theta = 0$, the system reduces to the well-known density-dependent incompressible liquid crystal flow [2–6]. Wen and Ding [7] proved the existence of local strong solutions. Fan, Gao and Guo [8] obtained the regularity criterion. The first aim of this paper is to prove some new regularity criteria of the above system, we will prove

Theorem 1.1. *Let $\nabla \rho_0 \in H^1 \cap W^{1,6}$, $u_0, \theta_0 \in H^2$, $d_0 \in H^3$ and $\frac{1}{C_1} \le \rho_0 \le C_1 < \infty$ and $\operatorname{div} u_0 = 0$. Let (ρ, u, θ, d) be the unique local strong solution to the problem (1.1)–(1.6). If u satisfies*

$$\int_0^T (\|u(t)\|_{L^\infty}^2 + \|\nabla u(t)\|_{L^\infty}) dt < \infty \tag{1.9}$$

with $0 < T < \infty$, then the solution (ρ, u, θ, d) can be extended beyond $T > 0$.

Theorem 1.2. *Let $\nabla \rho_0 \in H^1 \cap W^{1,6}$, $u_0, \theta_0, \nabla d_0 \in H^2$ and $\frac{1}{C_1} \le \rho_0 \le C_1$, $\operatorname{div} u_0 = 0$, and $|d_0| = 1$. Let (ρ, u, θ, d) be the unique local strong solution to the problem (1.1)–(1.4), (1.6) and (1.8). If u satisfied (1.9) and*

$$\nabla d \in L^2(0, T; \dot{B}^0_{\infty,\infty}) \tag{1.10}$$

with $0 < T < \infty$, then the solution (ρ, u, θ, d) can be extended beyond $T > 0$. Here $\dot{B}^0_{\infty,\infty}$ is the homogeneous Besov space.

Remark 1.1. The same regularity criterion (1.9) has been proved for the density-dependent MHD system [9,10].

When $u = 0$, the system (1.8) is the well-known harmonic heat flow.

Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, n be the outward unit normal to $\partial\Omega$. We consider the following 3D simplified nonisothermal liquid fluid flow

$$\partial_t u + u \cdot \nabla u + \nabla \pi - \operatorname{div}(\mu(\theta)(\nabla u + \nabla u^t)) = -\operatorname{div}(\nabla d \odot \nabla d), \tag{1.11}$$

$$\operatorname{div} u = 0, \tag{1.12}$$

$$\partial_t \theta + u \cdot \nabla \theta - \operatorname{div}(k(\theta)\nabla\theta) = 0, \tag{1.13}$$

$$\partial_t d + u \cdot \nabla d - \Delta d = d(1 - |d|^2) \quad \text{in } \Omega \times (0, \infty), \tag{1.14}$$

$$u = 0, \frac{\partial \theta}{\partial n} = 0, \frac{\partial d}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0, \infty), \tag{1.15}$$

$$(u, \theta, d)(\cdot, 0) = (u_0, \theta_0, d_0) \quad \text{in } \Omega \subset \mathbb{R}^3. \tag{1.16}$$

k is the heat conductivity, we will assume that k is a smooth function and satisfies

$$\frac{1}{C_1} \le k(\theta) \le C_1 \quad \text{when } |\theta| \le C_2 \tag{1.17}$$

for some positive constants $C_1 > 1$ and $C_2 > 0$.

Finally, we consider the following system: (1.11)–(1.13), (1.15), (1.16) and (1.8).

When $\theta = 0$, the system reduces to the well-known incompressible liquid crystal flow [11–17]. When d is a unit constant vector, the system reduces to the well-known Boussinesq system, Fan, Li and Nakamura [18] show a blow-up criterion. The aim of this paper is to prove some new regularity criteria, we will prove

Theorem 1.3. *Let $u_0 \in H_0^1 \cap H^2$, $\theta_0 \in H^2$, $d_0 \in H^3$ with $\operatorname{div} u_0 = 0$ in Ω, $\frac{\partial \theta_0}{\partial n} = 0$, $\frac{\partial d_0}{\partial n} = 0$ on $\partial\Omega$. Let μ and k satisfy (1.7) and (1.17). Let (u, θ, d) be a local strong solution to the problem (1.11)–(1.16). If u satisfies*

$$u \in L^r(0, T; L^p) \quad \text{with } \frac{2}{r} + \frac{3}{p} = 1 \text{ and } 3 < p < \infty \tag{1.18}$$

with $0 < T < \infty$, then the solution (u, θ, d) can be extended beyond $T > 0$.

Theorem 1.4. *Let $u_0 \in H_0^1 \cap H^2$, $\theta_0 \in H^2$, $d_0 \in H^3$ with $\operatorname{div} u_0 = 0$, $|d_0| = 1$ in Ω, $\frac{\partial \theta_0}{\partial n} = 0$, $\frac{\partial d_0}{\partial n} = 0$ on $\partial\Omega$. Let μ and k satisfy (1.7) and (1.17). Let (u, θ, d) be a local strong solution to the problem (1.11)–(1.13), (1.15), (1.16) and (1.8). If u and ∇d satisfy (1.18) and*

$$\nabla d \in L^s(0, T; L^q) \quad \text{with } \frac{2}{s} + \frac{3}{q} = 1 \text{ and } 3 < q \le \infty \tag{1.19}$$

with $0 < T < \infty$, then the solution (u, θ, d) can be extended beyond $T > 0$.

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