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Multiple positive solutions to a Kirchhoff type problem involving a critical nonlinearity

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ABSTRACT

In this paper, we investigate a class of Kirchhoff type problem involving a critical nonlinearity

$$-\left(1+b\int_{\Omega}|\nabla u|^2dx\right)\triangle u=\lambda u+|u|^4u, \quad u\in H_0^1(\Omega),$$

where $b > 0$, $\lambda > \lambda_1$, λ_1 is the principal eigenvalue of $(-\triangle, H_0^1(\Omega))$. With the help of the Nehari manifold, we obtain the multiplicity of positive solutions for λ in a small right neighborhood of λ_1 and prove that one of the solutions is a positive ground state solution, which is different from the result of Brézis–Nirenberg in 1983. This paper can be regarded as the complementary work of Naimen (2015).

1. Introduction and main result

In this paper, we are concerned with the multiplicity of positive solutions to the following Kirchhoff type problem:

$$\begin{cases} -\left(1+b\int_{\Omega}|\nabla u|^2dx\right)\triangle u=\lambda u+|u|^4u, & x\in\Omega,\\ u=0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, b is a positive constant, λ is a positive parameter, and the nonlinear growth of $|u|^4u$ reaches the Sobolev critical exponent since the critical exponent $2^*=6$ in three spatial dimensions.

Problem (1.1) is called nonlocal because of the presence of the term

$$-\left(1+b\int_{\Omega}|\nabla u|^2dx\right),$$

which implies that the equation in (1.1) is no longer a point-wise identity. This phenomenon provokes some mathematical difficulties, which make the study of such a class of problems particularly interesting. Problem (1.1) is related to the stationary analogue of the equation

$$\begin{cases} u_{tt}-\left(a+b\int_{\Omega}|\nabla u|^2dx\right)\triangle u=h(x,u), & x\in\Omega,\\ u=0, & x\in\partial\Omega. \end{cases} \tag{1.2}$$

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proposed by Kirchhoff [1] in 1883 as a generalization of the well-known D'Alembert wave equation for free vibrations of elastic strings. We have to point out that nonlocal problems also appear in other fields, such as biological systems, please refer to [2,3]. However, problem (1.2) received great attention only after Lions [4] proposed an abstract functional analysis framework for the problem.

When $b = 0$ in problem (1.1), it reduces to the classic semilinear elliptic problem which starts from the celebrated paper by Brézis and Nirenberg [5]. After that many authors are devoted to the investigations for a variety of elliptic equations with critical growth on bounded domain or whole space. Remarkably, (1.1) is a nonlocal problem which causes that the energy functional has totally different properties from the case $b = 0$, which makes the study of problem (1.1) particularly interesting.

The solvability or multiplicity of the Kirchhoff type problem with critical exponents has been paid much attention to various authors by means of the variational method, the genus theory, the fountain theorem, the Nehari manifold and the Ljusternik–Schnirelmann category theory, and so on, see [6–20]. In particular, $\Omega \subset \mathbb{R}^3$, for the following Kirchhoff type problem

$$\begin{cases} -\left(a + b\displaystyle\int_\Omega |\nabla u|^2 dx\right)\triangle u = \lambda h(x,u) + |u|^4 u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

if $h(x, u)$ is a continuous superlinear and subcritical nonlinearity on u, Alves et al. [6], Figueiredo [7] have studied the existence and multiplicity of positive solutions when the parameter λ is sufficiently large by the variational method and an appropriated truncated argument. If $h(x, u) = h(x)|u|^{q-1}u$ and $3 < q < 5$, Fan [8] has investigated the multiplicity results of positive solutions for $\lambda = 1$ by the Nehari manifold and the Ljusternik–Schnirelmann category theory. If $h(x, u) = h(x)$, Liu et al. [9] have obtained the existence and multiplicity of positive solutions in $\Omega = \mathbb{R}^3$ for small enough λ by using the variational method. Especially, when $h(x, u) = |u|^{q-1}u$, the above problem becomes

$$\begin{cases} -\left(a + b\displaystyle\int_\Omega |\nabla u|^2 dx\right)\triangle u = \lambda |u|^{q-1}u + |u|^4 u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

If $1 < q \leq 3$, Naimen [10] has obtained that there exists a constant $\lambda_0 \geq 0$ such that for all $\lambda > \lambda_0$, problem (1.3) has a positive solution by another truncation method. If $0 < q < 1$, problem (1.3) involves the concave and convex nonlinearities, Sun et al. [11] have obtained a positive solution for small enough λ by the Nehari manifold. If $-1 < q < 0$, the nonlinearities contain the singular term, Lei et al. [12] have got two positive solutions for small enough λ via the variational and perturbation methods.

Therefore, based on the above researches, we find that most of authors only think about the case $q \neq 1$. If $q = a = 1$, problem (1.3) becomes problem (1.1). Let λ_1 be the principal eigenvalue of $(-\triangle, H_0^1(\Omega))$, we have the following variational characterization

$$\lambda_1 := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|^2}{\int_\Omega |u|^2 dx}. \tag{1.4}$$

If $\lambda > \lambda_1$, it is well known that Brézis and Nirenberg [5] have proved that problem (1.1) has no solution if $b = 0$. However, Kirchhoff type elliptic problems involve a nonlocal perturbation which causes that the energy functional has totally different properties from the case $b = 0$. Furthermore, we delicately analyze the behavior of $\left(\int_\Omega |\nabla u|^2 dx\right)\triangle u$ and find that, in the competing of the term $\left(\int_\Omega |\nabla u|^2 dx\right)\triangle u$ and the critical term, the former may dominate the situation, which implies that it is likely to obtain the existence of solutions to problem (1.1). This observation indicates that some time the appearance of the term $\left(\int_\Omega |\nabla u|^2 dx\right)\triangle u$ is good. Recently, Naimen [13] has proved that if $\lambda > \lambda_1$, there exists a constant $b_0 = b_0(\lambda) > 0$ such that, for all $b \geq b_0$, problem (1.1) has a solution by the concentration compactness principle [21].

Motivated by the works described above, particularly, by the results in [8–13], we try to obtain the multiplicity of positive solutions and the existence of positive ground state solutions to problem (1.1) for λ in a small right neighborhood of λ_1 and all $b > 0$. Noting that Naimen [13] obtained only one solution to problem (1.1), we will illustrate the multiplicity of positive solutions. The main results can be described as follows.

Theorem 1.1. *For each $\lambda_1 < \lambda < \lambda_1 + \delta$, $\delta = \frac{\lambda_1 b^2 S^3}{8}$, problem (1.1) has at least one positive ground state solution, where S is the best Sobolev constant for the embedding of $H_0^1(\Omega)$ in $L^6(\Omega)$.*

Remark 1.1. It is easy to see that the Nehari manifold contains every nontrivial solutions of problem (1.1). Hence, the ground state solution can be obtained by finding the global minimizer of the Nehari manifold. Furthermore, we can confirm the ground state solution lying a certain part of the Nehari manifold.

Theorem 1.2. *There exists $0 < \delta^* < \delta$ such that, for each $\lambda_1 < \lambda < \lambda_1 + \delta^*$, problem (1.1) has at least two positive solutions.*

Remark 1.2. (1) We point out, as far as we are concerned, that this paper seems to be the first attempt to obtain the multiplicity of positive solutions to problem (1.1) for λ in a small right neighborhood of λ_1 on bounded domain. On