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Existence and multiplicity of solutions for a superlinear Kirchhoff-type equations with critical Sobolev exponent in $\mathbb{R}^N$ ☆

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ABSTRACT

In this article, we consider a class of superlinear Kirchhoff-type equations with critical growth

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)\Delta u = \mu u^{2^*-1}+\lambda k(x)u^{q-1}, \quad x\in\mathbb{R}^N, \\ u\in D^{1,2}(\mathbb{R}^N), \end{cases}$$

where $\lambda,\mu>0,\ N\geq 4,\ 2\leq q<2^*,\ 2^*=\frac{2N}{N-2},\ a,b\geq 0$ and $a+b>0$, k satisfies some conditions. By using the variational method, the multiplicity of positive solutions and infinitely many pairs of solutions are obtained.

1. Introduction and main results

In this paper, we consider the following Kirchhoff-type problems involving the critical Sobolev exponent

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u = \mu u^{2^*-1}+\lambda k(x)u^{q-1}, \quad x\in\mathbb{R}^N, \\ u\in D^{1,2}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N\geq 3, \mu, \lambda>0,\ 2\leq q<2^*,\ a,b\geq 0$ and $a+b\geq 0$. $2^*=\frac{2N}{N-2}$ is the critical Sobolev exponent for the embedding of $D^{1,2}(\mathbb{R}^N)$ into $L^s(\mathbb{R}^N)$ for every $s\in[1,2^*]$, where $D^{1,2}(\mathbb{R}^N)$ is a Sobolev space equipped with the norm $\|u\|=\left(\int_{\mathbb{R}^N}|\nabla u|^2dx\right)^{\frac{1}{2}}$ and the Lebesgue space $L^s(\mathbb{R}^N)$ with the norm $|u|_s=\left(\int_{\mathbb{R}^N}|u|^s dx\right)^{\frac{1}{s}}$. $k\in L^{\frac{2^*}{2^*-q}}(\mathbb{R}^N)$ is nonzero and nonnegative. When $a=0,\ b>0$ problem (1.1) is called degenerate, otherwise is called non-degenerate.

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It is well-known that the Kirchhoff-type equations have a strongly physical meaning. In 1883, the Kirchhoff-type problem was firstly proposed by Kirchhoff in [1] as a model given by the stationary analogue of equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.2}$$

where ρ, P_0, h, E, L are constants, which have the following meanings: ρ is the mass density, P_0 is the initial tension, h represents the area of the cross-section, E is the Young modulus of the material, and L is the length of the string. The above model is an extension of the classical D'Alembert's wave equation by taking into account the changes in the length of the string during the transverse vibrations. It is pointed out that problem (1.2) contains a nonlocal term $\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx$. It is worth noticing that problem (1.2) received much attention only after the work of Lions [2] where a function analysis framework was proposed to the problem.

Recently, the solvability or multiplicity of the Kirchhoff-type problem with critical exponent has been paid much attention by many authors, see [3–28]. In [3], Alves, Corrêa and Figueiredo considered the existence of solutions for the Kirchhoff-type problem with critical exponent. By using the variational method, they obtained a positive solution. To our best knowledge, this paper is the first result on the Kirchhoff-type problem with critical exponent. In [13,15,25], the concentration behavior of the positive solutions is considered. Liang and Shi obtained the existence of multiple soliton solutions for the Kirchhoff-type problem involving a modified term in [18]. In [19], Liang and Zhang investigated the existence and multiplicity of solutions for the singularly perturbed Kirchhoff-type problems.

Particularly, Liu, Liao and Tang in [20] considered problem (1.1) with $\mu=q=1$ and $k\in L^{\frac{2^*}{2^*-1}}(\mathbb{R}^N)\setminus\{0\}$. The existence and nonexistence of positive solutions are got under some appropriate conditions for N, a, b and λ by the variational method. Naimen in [21] studied problem (1.1) with $N=4$ in a bounded domain. By the variational method and the concentration compactness argument, the author obtained the existence of positive solutions. Very recently, when $N=3$, Lei et al. studied problem (1.1) with $\mu=1$ and $2<q<6$ in [16], where k satisfies the following conditions

(k_1) $k\in L^{\frac{6}{6-q}}(\mathbb{R}^3)$ and $k(x)\geq 0$ for any $x\in\mathbb{R}^3$ and $k\not\equiv 0$;
(k_2) There exist $x_0\in\mathbb{R}^3$ and $\delta,\rho_1>0$ such that $k(x)\geq\delta|x-x_0|^{-\beta}$ for $|x-x_0|<\rho_1$ and $0<\beta<3$.

By the variational method and concentration-compactness principle, they got a positive ground state solution for problem (1.1).

Inspired by [16,21], it is natural to ask whether problem (1.1) with $N\geq 4$ has the existence of positive solutions. In the present note, we give a positive answer by the variational method. We obtain two positive solutions for problem (1.1) with $2<q<2^*$. To our best knowledge, the multiplicity of positive solutions for the superlinear Kirchhoff-type problems with critical exponent is the first result up to now. Moreover, when $N=4,\ a=0,\ b>0$, we obtain infinitely many pairs of distinct solutions for problem (1.1) by a critical point theorem. From now on, we always assume $N\geq 4$.

We define the energy functional corresponding to problem (1.1) by

$$I(u)=\frac{a}{2}\|u\|^2+\frac{b}{4}\|u\|^4-\frac{\mu}{2^*}\int_{\mathbb{R}^N}|u|^{2^*}dx-\frac{\lambda}{q}\int_{\mathbb{R}^N}k(x)|u|^q dx,\quad \forall u\in D^{1,2}(\mathbb{R}^N).$$

Obviously, I is well defined and $I\in C^1(D^{1,2}(\mathbb{R}^N),\mathbb{R})$ and has the derivative given by

$$\langle I'(u),v\rangle=(a+b\|u\|^2)\int_{\mathbb{R}^N}(\nabla u,\nabla v)dx-\mu\int_{\mathbb{R}^N}|u|^{2^*-2}uvdx-\lambda\int_{\mathbb{R}^N}k(x)|u|^{q-2}uvdx$$

for all $u,v\in D^{1,2}(\mathbb{R}^N)$. As well known, there exists a one to one correspondence between the solutions of problem (1.1) and the critical points of I on $D^{1,2}(\mathbb{R}^N)$.

We denote by S the best constant for the Sobolev embedding $D^{1,2}(\mathbb{R}^N)\hookrightarrow L^{2^*}(\mathbb{R}^N)$, namely

$$S:=\inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}}\frac{\int_{\mathbb{R}^N}|\nabla u|^2dx}{\left(\int_{\mathbb{R}^N}|u|^{2^*}dx\right)^{\frac{2}{2^*}}}, \tag{1.3}$$

and let Λ be a constant, by

$$\Lambda=\left(\frac{2a}{4-2^*}\right)^{\frac{4-2^*}{2}}S^{\frac{2^*}{2}}\left(\frac{2b}{2^*-2}\right)^{\frac{2^*-2}{2}},\quad N\geq 5.$$

Now our main results can be described as follows:

Theorem 1.1. *Let $a,b>0,\ 2<q<2^*,\ N\geq 4$ and*

$$0<\mu<\begin{cases} bS^2, & N=4,\\ \Lambda, & N\geq 5.\end{cases} \tag{1.4}$$

Assume $k\in L^{\frac{2^*}{2^*-q}}(\mathbb{R}^N)$ with $k\geq 0$ and $k\not\equiv 0$, then there exists $\lambda^*>0$ such that problem (1.1) has least two positive solutions for all $\lambda>\lambda^*$.