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A space–time fully decoupled wavelet Galerkin method for solving two-dimensional Burgers' equations

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ABSTRACT

A space–time fully decoupled formulation for solving two-dimensional Burgers' equations is proposed based on the Coiflet-type wavelet sampling approximation for a function defined on a bounded interval. By applying a wavelet Galerkin approach for spatial discretization, nonlinear partial differential equations are first transformed into a system of ordinary differential equations, in which all matrices are completely independent of time and never need to be updated in the time integration. Finally, the mixed explicit–implicit scheme is employed to solve the resulting semi-discretization system. By numerically studying three widely considered test problems, results demonstrate that the proposed method has a much better accuracy and a faster convergence rate than many existing numerical methods. Most importantly, the study also indicates that the present wavelet method is capable of solving the two-dimensional Burgers' equation at high Reynolds numbers.

1. Introduction

The two-dimensional Burgers' equation, which incorporates both nonlinear convection and viscous diffusion, can be written into the conservative form [1–10]

$$\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial u^2}{\partial x}+\frac{1}{2}\frac{\partial u^2}{\partial y}=\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{1}$$

in which $u(x, y, t)$ is the dependent variable, and Re is referred to as the Reynolds number characterizing the size of viscosity. This hyperbolic–parabolic equation has been widely used for modeling various physical phenomena, such as the turbulence, the boundary layer, the shock wave propagation, and the traffic flow [1–10].

The Burgers' equation (1) is very similar to the Navier–Stokes equations and a shock wave may also arise at high Reynolds numbers [8–12]. Moreover, it is also one of a very few nonlinear partial differential equations, which can be solved exactly under a set of appropriate initial and boundary conditions [1–12]. Therefore, the Burgers' equation (1) was frequently used as a numerical test for various numerical methods [1–27], such as the finite difference method [1–6], the Galerkin

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type method [7–10], the finite element method [11–15], the collocation method [16–18], the Lattice Boltzmann method [19–22], and the differential quadrature method [23–25]. These methods are effective for solving the two-dimensional Burgers' equation under certain conditions. However, most of them will encounter severe difficulties in solving Burgers' problems with a high Reynolds number. For example, Duan and Liu [21] proposed a special lattice Boltzmann model to simulate two-dimensional unsteady Burgers' equation. A good approximate solution with maximum absolute error $O(10^{-4})$ can be obtained under spatial grid 20×20 for Reynolds number $Re = 1$. However for $Re = 100$, the maximum absolute error of the numerical solution has sharply increased to $O(10^{-2})$ even using more fine spatial grid 200×200. Islam et al. [17] studied the effectiveness of the global meshless collocation method, the Legendre wavelet collocation method and the Haar wavelet collocation method for solving the two-dimensional Burgers' equation under various Reynolds numbers. Their results demonstrate that these methods can effectively handle the Burgers' equation at low Reynolds numbers; however, solutions achieved from all these three methods are unstable and highly distorted for high Reynolds numbers.

Moreover, when the conventional Galerkin type method and finite element method are employed to solve directly the Burgers' equation, the matrix generated in the spatial discretization of nonlinear convection term will be dependent on the time-dependent unknown vector [7–15]. Thus, this matrix obtained by numerical integration must be recalculated at each time step, thereby consuming considerable computing resources. For example, Fletcher [13] applied the conventional finite element method to transform equation (1) into a system of nonlinear ordinary differential equations:

$$\mathbf{M}d\mathbf{U}/dt = [\mathbf{C}/Re - \mathbf{B}(\mathbf{U})]\mathbf{U}. \tag{2}$$

In Eq. (2), $\mathbf{U}(t)$ is the time-dependent unknown column vector, and matrices $\mathbf{M} = \int_\Omega \mathbf{\Psi}^{\mathrm{T}}\mathbf{\Psi}d\Omega$, $\mathbf{C} = \int_\Omega (d\mathbf{\Psi}^{\mathrm{T}}/dx d\mathbf{\Psi}/dx + d\mathbf{\Psi}^{\mathrm{T}}/dy d\mathbf{\Psi}/dy)d\Omega$, and $\mathbf{B}(\mathbf{U}) = \int_\Omega \mathbf{\Psi}^{\mathrm{T}}(\mathbf{\Psi}\mathbf{U})(d\mathbf{\Psi}/dx + d\mathbf{\Psi}/dy)d\Omega$, where the vector $\mathbf{\Psi}(x)$ represents the shape functions in spatial domain Ω. It can be seen from Eq. (2) that the matrix $\mathbf{B}(\mathbf{U})$ from the spatial discretization of nonlinear convection term $u\partial u/\partial x + u\partial u/\partial y$ relies explicitly on the unknown vector $\mathbf{U}(t)$, and should be updated at each time step in the subsequent time integration by using the finite difference scheme [13]. The similar problem is also encountered in the boundary element method proposed by Chino and Tosaka [26]. In fact, repeated recalculations of the matrix from the spatial discretization of nonlinear term can be regarded as re-performing the spatial discretization at each time step. Therefore, the decoupling between spatial and temporal discretizations in these existing Galerkin type method and finite element method [7–15] is incomplete, because they cannot divide the solution procedure into two completely separate processes. As such, additional computational cost is needed to update the matrix at each time step, which is generated in the spatial discretization of nonlinear convection term in Eq. (1). In order to alleviate the above problem, Zhang et al. [9] introduced an increment dimensional technique to obtain a time-independent matrix $\bar{\mathbf{B}}$, which represents the spatial discretization of nonlinear convection term in Eq. (1). However, the dimension of $\bar{\mathbf{B}}$ has increased to $2N \times 2N$ where N is the number of degree of freedom, thereby increasing computational cost and storage space.

In this study, we combine a wavelet Galerkin technique with the mixed explicit–implicit scheme [28–30] to solve the two-dimensional Burgers' equation with the Dirichlet and periodic boundary conditions. By using a modified wavelet Galerkin method for the spatial discretization, the nonlinear partial differential equation (1) is transformed into a system of nonlinear ordinary differential equations, which is further solved applying the mixed explicit–implicit scheme. Most importantly, all matrices in the resulting semi-discretization system are $N \times N$ dimensional constant matrices and need not to be updated in the subsequent time integration, i.e., a fully decoupling between spatial and temporal discretizations is achieved in the proposed wavelet method. Finally, by studying three widely considered problems with various Reynolds numbers, the present solutions and those obtained by using many existing numerical methods are compared to demonstrate the effectiveness of the proposed space–time fully decoupled wavelet formulation.

2. Sampling approximation of an interval-bounded L^2-function

On the basis of the wavelet multiresolution analysis [31] and our previous work [32–35], a function $f(x) \in L^2(R)$ can be approximated by

$$f(x) \approx P^j f(x) = \sum_{k\in Z} f\left(\frac{k}{2^j}\right)\phi(2^j x - k + M_1) \tag{3}$$

where $M_1 = \int x\phi(x)dx$ is the first order moment of the Coiflet-type orthogonal scaling function $\phi(x)$, and j is the decomposition level. The accuracy of the wavelet approximation (3) was estimated as [31,34]

$$\left\| \frac{d^n f(x)}{dx^n} - \frac{d^n P^j f(x)}{dx^n} \right\|_{L^2} \le C 2^{-j(\lambda-n)} \tag{4}$$

in which C is a constant, λ is the number of vanishing moment of the wavelet function corresponding to scaling function $\phi(x)$, and non-negative integer $n < \lambda$. In the present study, the scaling function $\phi(x)$ with $M_1 = 7$ and $\lambda = 6$ [32] is adopted.

However, when using Eq. (3) to approximate a function defined on a bounded interval, some extra treatments should be employed to avoid instability problems, because the original wavelet approximation (3) is suitable for functions defined on the whole real line [31,36].