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A singularly perturbed problem with two parameters in two dimensions on graded meshes

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ABSTRACT

A numerical approximation of a convection–reaction–diffusion problem by standard bilinear finite elements is considered. Using Duran–Lombardi and Duran–Shishkin type meshes we prove first order error estimates in an energy norm. Numerical examples confirm our theoretical results and show smaller errors compared to the well-known Shishkin mesh.

1. Introduction

Modeling of physical phenomena often involves boundary value problems with small parameters. Nevertheless, not much is known about robust numerical methods for solving two-parameter singularly perturbed problems. This kind of problem arises in chemical flow reactor theory [1], and in the case of boundary layers controlled by suction of some fluid [2].

Standard numerical methods are inefficient for singularly perturbed problems. The main goal in the construction of numerical methods for these problems is acquiring their uniform convergence (robustness) with respect to all perturbation parameters. If u is the solution of a singularly perturbed problem, and u^n its numerical approximation obtained by a numerical method with n degrees of freedom, then the numerical method is uniformly convergent with respect to the perturbation parameters in the norm $\|\cdot\|$, if

$$\|u-u^n\| \le \vartheta(n) \quad \text{for } n \ge n_0,$$

with the function ϑ satisfying $\lim_{n\to\infty} \vartheta(n) = 0$, [3].

Here, we consider a finite element method for the following singularly perturbed elliptic two-parameter problem on the unit square $\Omega = (0,1)\times(0,1)$

$$\begin{aligned} Lu := -\varepsilon_1 \Delta u + \varepsilon_2 b(x) u_x + c(x) u = f(x,y), \quad (x,y)\in\Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

with

$$b(x) \ge b_0 > 0, \qquad c(x) \ge c_0 > 0, \quad x\in(0,1), \tag{2}$$

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where b, c and f are sufficiently smooth functions, b_0, c_0 are constants, $0 < \varepsilon_1, \varepsilon_2 \ll 1$ are small perturbation parameters and f satisfies the compatibility conditions

$$f(0,0) = f(0,1) = f(1,0) = f(1,1) = 0. \tag{3}$$

Under these assumptions there exists a classical solution u of the problem (1) such that $u \in C^{3,\alpha}(\bar{\Omega})$ for $\alpha \in (0,1)$, [4]. We also assume

$$c(x) - \frac{1}{2}\varepsilon_2 b'(x) \geq \gamma > 0, \quad x \in (0,1), \tag{4}$$

for some constant γ. It is known that the problem (1) is characterized by exponential layers at $x = 0$ and $x = 1$, parabolic layers at $y = 0$ and $y = 1$ and corner layers at four corners of Ω, [5]. The width of exponential layers depends on the relation between ε_1 and ε_2. For $\varepsilon_2 = 0$, the problem (1) is a reaction–diffusion problem as opposed to $\varepsilon_2 = 1$, when it becomes a convection–diffusion problem. In this paper we consider problems where $0 < \varepsilon_2 \ll 1$, for arbitrary relation between ε_1 and ε_2. The problem (1) is considered in detail in [5,6] and numerically solved using finite element method on a piecewise uniform Shishkin mesh. Earlier, the similar problem was solved by the same method in [7]. The mesh construction and error analysis in our paper are based on a decomposition of the solution from [5]. In [8–12], the authors analyzed different type of elliptic convection–diffusion equations with two parameters.

In this paper, we give a construction of layer-adapted meshes called Duran–Lombardi and Duran–Shishkin type meshes and prove uniform convergence in an energy norm. These graded meshes seem to be an interesting alternative to the well-known Shishkin meshes. From numerical experiments given in [13], the graded meshes procedure appears to be more robust in the sense that numerical results are not strongly affected by variations of parameters defining the meshes. Moreover, the graded meshes have some desirable properties that the Shishkin meshes do not have. When we approximate a singularly perturbed problem with an a priori adapted mesh, it is natural to expect that a mesh designed for some value of the small parameter will also work well for larger values of it. In this regard the Duran–Lombardi and Duran–Shishkin type meshes have better behavior.

This paper is a nontrivial extension of the work on a one-dimensional singularly perturbed two-parameter problem [14]. Due to the presence of parabolic and corner layers, the error analysis for the problem in two dimensions is more advanced. It contains new technical details and requires careful inspection of the layer components on different subdomains of Ω.

The rest of the paper is organized as follows. In Section 2, some solution properties and solution decomposition is given. Also, we present a finite element method for our problem. Graded meshes are introduced in Section 3. Error estimates on Duran–Lombardi and Duran–Shishkin meshes are presented in the next section. Based on interpolation and discretization errors, we prove a robustness of the Galerkin finite element method in an energy norm. The last section contains some numerical results.

Notation 1. *Throughout the paper, C will denote a generic positive constant independent of perturbation parameters ε_1, ε_2 and of the number of degrees of freedom n. For a set $D \subset \mathbb{R}$, standard notation for Banach spaces $L^p(D)$, Sobolev spaces $W^{k,p}(D)$, $H^k(D) = W^{k,2}(D)$, norms $\|\cdot\|_{L^p(D)}$ and seminorms $|\cdot|_{H^k(D)}$ are used.*

2. Solution properties

In order to describe the exponential layers we introduce characteristic equation for (1)

$$-\varepsilon_1 r^2(x) + \varepsilon_2 b(x) r(x) + c(x) = 0,$$

which has two real solutions $r_0(x) < 0$ and $r_1(x) > 0$. Let

$$\mu_0 = -\max_{x\in[0,1]} r_0(x), \qquad \mu_1 = \min_{x\in[0,1]} r_1(x),$$

namely

$$\mu_{0,1} = \min_{x\in[0,1]} \frac{\mp\varepsilon_2 b(x) + \sqrt{\varepsilon_2^2 b^2(x) + 4\varepsilon_1 c(x)}}{2\varepsilon_1}.$$

Particularly, in numerical examples, μ_0 and μ_1 are calculated in the following way

$$\mu_0 = \frac{-\varepsilon_2 B + \sqrt{\varepsilon_2^2 B^2 + 4\varepsilon_1 c_0}}{2\varepsilon_1}, \qquad \mu_1 = \frac{\varepsilon_2 b_0 + \sqrt{\varepsilon_2^2 b_0^2 + 4\varepsilon_1 c_0}}{2\varepsilon_1}, \tag{5}$$

where $B = \max_{x\in[0,1]} b(x)$. Further, from (5) we obtain that $\mu_0 > 1$ is valid for

$$\max\{\varepsilon_1, \varepsilon_2^2\} < \frac{4c_0^2}{(B + \sqrt{B^2 + 4c_0})^2}. \tag{6}$$

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