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Determination of a time-dependent convolution kernel from a boundary measurement in nonlinear Maxwell's equations

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ABSTRACT

A nonlinear hyperbolic Maxwell equation with an unknown solely time dependent convolution kernel is studied. The missing kernel is recovered from an additional normal component measurement over the whole boundary. The existence of a solution to this inverse problem is shown. Moreover a constructive algorithm for approximations is designed and its convergence is established. Uniqueness is proved for a regular solution. Theoretical results are supported by a numerical experiment.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded and connected domain. We assume that Ω is either smooth $\Omega \in C^{1,1}$ or convex. The symbol $\boldsymbol{\nu}$ stands for the outer normal vector associated with the boundary $\partial\Omega = \Gamma$. We adopt the following notations $\nabla\times = \text{curl}$, $\nabla\cdot = \text{divergence}$, ∂_t means time derivative and $\nabla = \text{gradient}$.

The starting point for the modeling of electromagnetic fields are the classical Maxwell's equations (see [1,2]), which consist of four (Maxwell–Ampere's, Gauss electric, Faraday's and Gauss magnetic) laws

$$\begin{aligned} \nabla\times\boldsymbol{H} - \partial_t\boldsymbol{D} &= \boldsymbol{J} + \boldsymbol{J}^{app}, & \nabla\cdot\boldsymbol{D} &= \rho, \\ \partial_t\boldsymbol{B} + \nabla\times\boldsymbol{E} &= \boldsymbol{0}, & \nabla\cdot\boldsymbol{B} &= 0, \end{aligned} \tag{1}$$

where $\boldsymbol{D}$ denotes the electric displacement, $\boldsymbol{B}$ stands for the magnetic induction, $\boldsymbol{H}$ and $\boldsymbol{E}$ are the magnetic and electric field respectively, $\boldsymbol{J}$ is the total current density and ρ is the density of electrical charge. The symbol $\boldsymbol{J}^{app}$ models a source term.

Physical laws (1) with the four unknowns $\boldsymbol{B}, \boldsymbol{D}, \boldsymbol{E}, \boldsymbol{H}$ are usually accompanied by constitutive relations of the type

$$\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{E}, \boldsymbol{H}), \qquad \boldsymbol{B} = \boldsymbol{B}(\boldsymbol{E}, \boldsymbol{H}),$$

where the exact form of the relationships depends on the situation under consideration. The dependencies can be linear (in linear materials) or nonlinear (in superconductors, nonlinear optics,...). Strictly speaking, from a physical point of view relations may be hereditary. In such a situation the present values of solutions depend on their past behavior. This may

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be expressed using a memory term in the form of a time integral. Applications can be found in chiral media [3], metamaterials [4,5] or polarized media [6]. The authors of [7] have considered a nonlinear memory effect for polarization $\boldsymbol{P}$ of the type

$$\boldsymbol{P}(t) = (g * [\boldsymbol{E} + \boldsymbol{f}(\boldsymbol{E})])(t).$$

Here the symbol $*$ stands for the usual convolution in time, namely $(K * \boldsymbol{u}(\boldsymbol{x}))(t) = \int_0^t K(t-s)\boldsymbol{u}(\boldsymbol{x}, s)\, \mathrm{d}s$. The formulation from [7] can be interpreted as a generalization of the Debye or Lorentz polarization models in the sense that the polarization dynamics is driven by a nonlinear function of the electric field.

In our paper we adopt a generalized Ohm's law of the following form

$$\boldsymbol{J} = \sigma * \boldsymbol{E} - 1 * \boldsymbol{g}(\boldsymbol{E}).$$

Further we assume that

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}$$

with a constant absolute permittivity ε, and a nonlinear magnetic material, i.e.

$$\boldsymbol{B} = \mu\,(\boldsymbol{H} - 1 * \boldsymbol{f}(\boldsymbol{E}))$$

with a positive variable permeability μ. Elimination of $\boldsymbol{H}$ in (1) leads to

$$\varepsilon \boldsymbol{E}_{tt} + (\sigma * \boldsymbol{E})_t + \nabla \times \left(\frac{1}{\mu} \nabla \times \boldsymbol{E} \right) = \boldsymbol{g}(\boldsymbol{E}) + \nabla \times \boldsymbol{f}(\boldsymbol{E}) - \boldsymbol{J}_t^{app}.$$

The conductivity term σ is assumed to be separable, i.e.

$$\sigma(\boldsymbol{x}, t) = \alpha(t)\tilde{\sigma}(\boldsymbol{x}),$$

where the given $\tilde{\sigma}(\boldsymbol{x})$ describes the heterogeneity of the material. We assume that $\tilde{\sigma}$ is constant along Γ with $\tilde{\sigma}|_\Gamma = \sigma^\Gamma$. This means that the possible inhomogeneity is an interior one. The hereditary weight $\alpha(t)$ is unknown and is has to be determined. We consider the following boundary condition modeling a perfect contact

$$\boldsymbol{E} \times \boldsymbol{\nu} = \boldsymbol{0} \quad \text{on } \Gamma \tag{2}$$

and the initial data

$$\boldsymbol{E}(\boldsymbol{x}, 0) = \boldsymbol{E}_0(\boldsymbol{x}), \qquad \boldsymbol{E}_t(\boldsymbol{x}, 0) = \boldsymbol{V}_0(\boldsymbol{x}). \tag{3}$$

The inverse problem (IP) is to find a couple $\{\boldsymbol{E}(\boldsymbol{x}, t), \alpha(t)\}$. The missing data function $\alpha(t)$ will be recovered by means of the following measurement along Γ

$$\int_\Gamma \boldsymbol{E} \cdot \boldsymbol{\nu}\, \mathrm{d}\gamma = m(t), \quad \text{(normal component measurement)}. \tag{4}$$

For ease of explanation we set $\varepsilon = 1$ and we omit $\boldsymbol{J}^{app}$ in order to enhance the readability of the manuscript.[1] Then the governing PDE reads as

$$\boldsymbol{E}_{tt} + \tilde{\sigma}\,(\alpha * \boldsymbol{E})_t + \nabla \times \left(\frac{1}{\mu} \nabla \times \boldsymbol{E} - \boldsymbol{f}(\boldsymbol{E}) \right) = \boldsymbol{g}(\boldsymbol{E}). \tag{5}$$

An integral overdetermination is frequently used in various inverse problems (IPs) for evolutionary problems, cf. [8–10] and the references therein. This appeared mostly for diffusion processes as in [8,11–15].

Identification of missing memory kernels in evolutionary PDEs was implemented in 1990's by Russian and Italian schools. This started the research in this area, cf. [16–26]. The paper [16] studies a linear hyperbolic equation without a damping term. Here, the time convolution operator acts on the Laplacian of solution. The unknown data are revealed from a point measurement. No constructive algorithm for finding a solution for this IP is presented. The paper [17] deals with identification of convolution kernels in abstract linear hyperbolic integro-differential problems. The local solvability in time of this IP is shown. There is no constructive algorithm for recovery of missing convolution kernel. The article [18] is devoted to a one dimensional linear hyperbolic integro-differential problem. The error estimates (for a numerical scheme based on finite differences with dependent time and space mesh-steps) are derived under high regularity of solution. [19] presents a nice study of properties of Dirichlet-to-Neumann maps for memory reconstruction for linear settings. In [23] a global in time existence and uniqueness result for an inverse problem arising in the theory of heat conduction for materials with memory has been studied. The paper [26] derives some local and global in time existence results for the recovery of memory kernels.

A constructive and very interesting technique for recovery of missing convolution kernels has been developed in [27–29] for scalar parabolic and hyperbolic equations. Here the additional measurement was a space integral of the solution over

[1] The term $\boldsymbol{J}^{app}$ can be handled in a standard way adopting suitable regularity assumptions, therefore we skip it.