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Finite element approximation of convection–diffusion problems using an exponentially graded mesh

Christos Xenophontos [a,*], Sebastian Franz [b], Lars Ludwig [c]

[a] Department of Mathematics and Statistics, University of Cyprus, P.O. Box 20537, 1678 Nicosia, Cyprus
[b] Technische Universität Dresden, Institut für Numerische Mathematik, 01062 Dresden, Germany
[c] Institut für Wissenschaftliches Rechnen, Technische Universität Dresden, 01062 Dresden, Germany

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ABSTRACT

We present the analysis of an h version Finite Element Method for the approximation of the solution to convection–diffusion problems. The method uses piece-wise polynomials of degree $p \geq 1$, defined on an *exponentially graded* mesh, optimally constructed for the approximation of exponential layers. We consider a model convection–diffusion problem, posed on the unit square and establish robust, optimal convergence rates in the energy and in the maximum norm. We also present the results of some numerical computations that illustrate our theoretical findings and compare the proposed method with others found in the literature.

1. Introduction

Singularly perturbed problems have been studied for a number of decades (see [1,2] and the references therein) as they arise in a variety of applications. One family of such problems are convection-dominated equations, in which the following two difficulties arise: stability of discretizations and the presence of *boundary layers* in the solution. In the context of the Finite Element Method (FEM), the robust approximation of boundary layers requires either the use of the h version on non-uniform, layer adapted meshes (such as the Shishkin [3] or Bakhvalov [4] mesh), or the use of the high order p and hp versions on appropriately designed (variable) meshes [5]. In both cases, the *a priori* knowledge of the position of the layers is taken into account and mesh-degree combinations can be chosen for which uniform error estimates can be established (see, e.g., [6]). Among the various layer adapted meshes that have been proposed in the literature over the years, one finds the non-uniform, layer adapted, *exponentially graded* mesh, which was constructed by optimizing certain upper bounds on the error (see [7] for details). The finite element analysis on this mesh appears in [8] where robust, optimal rates of convergence were proved for one-dimensional reaction–diffusion and convection–diffusion problems. The purpose of this article is to extend the results of [8] to a two-dimensional convection–diffusion problem posed on a square, under sufficient regularity assumptions (which exclude the presence of corner singularities in the solution). The proposed method not only yields the optimal rate of convergence, but it does so without the assumption $\varepsilon \leq CN^{-1}$, as is done in most other methods; here ε is the singular perturbation parameter, N is the number of meshpoints and C is a positive constant independent of ε and N. In addition, in numerical tests it outperforms the equally optimal Bakhvalov–Shishkin mesh (see [8] for one-dimensional results and Section 5 ahead).

* Corresponding author.
E-mail address: xenophontos@ucy.ac.cy (C. Xenophontos).

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The rest of the paper is organized as follows: in Section 2 we present the model problem and its regularity. The discretization using the exponentially graded mesh is presented in Section 3 and in Section 4 we present our main result of uniform, optimal convergence. Section 5 shows the results of some numerical computations that illustrate the theoretical findings and Section 6 gives our conclusions.

With $\Omega \subset \mathbb{R}^2$ a bounded open set with Lipschitz boundary $\partial\Omega$ and measure $|\Omega|$, we will denote by $C^k(\Omega)$ the space of continuous functions on Ω with continuous derivatives up to order k and by $C^{k,\alpha}(\Omega)$ we will denote the space of functions whose derivatives up to order k are Hölder continuous with exponent α. We will use the usual Sobolev spaces $W^{k,m}(\Omega)$ of functions on Ω with $0, 1, 2, \ldots, k$ generalized derivatives in $L^m(\Omega)$, equipped with the norm and seminorm $\|\cdot\|_{k,m,\Omega}$ and $|\cdot|_{k,m,\Omega}$, respectively. When $m = 2$, we will write $H^k(\Omega)$ instead of $W^{k,2}(\Omega)$, and for the norm and seminorm, we will write $\|\cdot\|_{k,\Omega}$ and $|\cdot|_{k,\Omega}$, respectively. We will also use the space

$$H_0^1(\Omega) = \left\{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\right\}.$$

The norm of the space $L^\infty(\Omega)$ of essentially bounded functions is denoted by $\|\cdot\|_{\infty,\Omega}$. Finally, the notation "$a \lesssim b$" means "$a \leq Cb$" with C being a generic positive constant, independent of any discretization or singular perturbation parameters.

2. The model problem and its regularity

We consider the following convection–diffusion boundary value problem (BVP): Find $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ such that

$$-\varepsilon\Delta u + \boldsymbol{b}^T\nabla u + cu = f \quad \text{in } \Omega = (0,1)^2, \tag{1a}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1b}$$

where

$$\boldsymbol{b}^T = [b_1(x,y), b_2(x,y)] \geq [\beta_1, \beta_2] > 0 \quad \text{on } \Omega,$$

$\varepsilon \in (0, 1]$ is a small parameter, b_1, b_2, c, f are sufficiently smooth functions and β_1, β_2 are constants. We will assume, as usual, that

$$c - \frac{1}{2}\text{div}\,\boldsymbol{b} > 0, \quad c \geq 0 \text{ on } \overline{\Omega}$$

and our problem will have a unique (weak) solution $u \in H_0^1(\Omega)$ for all $f \in L^2(\Omega)$.

The analysis will be performed under the assumption that the solution does not contain any corner singularities. This will happen if certain compatibility conditions are imposed on the data (see, e.g., [9]). Under such an assumption, the solution u will admit a decomposition (see, e.g., [6]) into a smooth part and a boundary layer part. Bounds on the derivatives of each part are available only in certain cases (see, e.g., [10]). To avoid corner singularities and focus on the boundary layers, we will make the following assumption:

Assumption 1. The BVP (1) has a classical solution $u \in C^{q+1,\alpha}\left(\overline{\Omega}\right)$, $q \geq 2$ for some $\alpha \in (0, 1)$, which can be decomposed as

$$u = S + E_1 + E_2 + E_{12}, \tag{A1}$$

where for all $(x, y) \in \overline{\Omega}$ there holds

$$\left|\frac{\partial^{i+j}S}{\partial x^i \partial y^j}(x,y)\right| \lesssim 1, \tag{A2}$$

$$\left|\frac{\partial^{i+j}E_1}{\partial x^i \partial y^j}(x,y)\right| \lesssim \varepsilon^{-i}e^{-\beta_1(1-x)/\varepsilon}, \tag{A3}$$

$$\left|\frac{\partial^{i+j}E_2}{\partial x^i \partial y^j}(x,y)\right| \lesssim \varepsilon^{-j}e^{-\beta_2(1-y)/\varepsilon} \tag{A4}$$

and

$$\left|\frac{\partial^{i+j}E_{12}}{\partial x^i \partial y^j}(x,y)\right| \lesssim \varepsilon^{-(i+j)}e^{-(\beta_1(1-x)+\beta_2(1-y))/\varepsilon}, \tag{A5}$$

for $0 \leq i + j \leq q$.

The above was proven for the case $q = 2$, $\Omega = (0, 1)^2$ in [11] and the precise compatibility of the data was given. See also [10] for a similar decomposition that includes corner singularities. Here, we assume that our problem has a unique solution that satisfies Assumption 1 for $q \geq 2$.