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Multiscale direction-splitting algorithms for parabolic equations with highly heterogeneous coefficients☆

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ABSTRACT

In this paper we discuss two methods for upscaling of highly heterogeneous data for parabolic problems in the context of a direction splitting time approximation. The first method is a direct application of the idea of Jenny et al. (2003) in the context of the direction splitting approach. The second method devises the approximation from the Schur complement corresponding to the interface unknowns of the coarse grid, by applying a proper L^2 projection operator to it. The spatial discretization employed in this paper is based on a MAC finite volume stencil but the same approach can be implemented within a proper finite element discretization. A key feature of the present approach is that it can extend to 3D problems with very little computational overhead. The properties of the resulting approximations are demonstrated numerically on some benchmark coefficient data available in the literature.

1. Introduction

The advances in the measurement and characterization of permeability in natural reservoirs nowadays lead to reservoir problems on the finest scale that are three dimensional and computationally demanding. This is why in the recent years, several multiscale techniques have been proposed in the literature that allow to coarsen the problem resolution but still retain some essential features of the original solution at the fine scale. These techniques are based most often on some variant of a finite element spatial discretization: the multiscale Galerkin finite element method [1–4], mixed multiscale finite element method (e.g. [5,6]), mortar multiscale methods (e.g. [7]), and variational multiscale method (e.g. [8]). For recent attempt to generalize these approaches within the framework of the Discontinuous Galerkin interior penalty method the reader is referred to [9]. Although it seems that the discretization preferred by the oil industry is based on the classical finite volumes (MAC) stencil, there are surprisingly few publications on multiscale finite volume methods, perhaps the most notable being the one proposed in [10]. From the most general point of view all these methods attempt to devise a more representative discretization space at the coarse scale by solving certain local boundary (see e.g. [10,2]) or eigenvalue (see e.g. [11,9]) problems on each coarse grid cell. These local problems are suitable for implementation on parallel clusters and, if the coefficients of the problem are independent of the solution (or time in the parabolic case), they need to be solved only once at the beginning of the computation (often done offline). Nevertheless, in the 3D case, although being significantly

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simpler than the fine-scale problem, the solution of the local problems can be a serious challenge. This is particularly true for methods based on approximation of some of the eigenfunctions of the local problems since the eigenvalue problems are computationally expensive. Therefore, for realistic applications it is important to minimize the cost of solving the local problems. Besides, taking into account the size of the reservoirs, the solution of the coarse-scale problem itself can also be a serious challenge. In this article we are exploring the possibility to significantly reduce the upscaling cost and the cost of the solution of the coarse-scale problem by employing splitting schemes. An additional benefit of the use of such schemes is that the resulting solvers are naturally parallelizable.

The most interesting subsurface flows involve a parabolic rather than an elliptic subproblem for the pressure. The multiscale finite volume method of [10] has been recently integrated within a multiphase flow model that requires the solution of a nonlinear parabolic equation for the pressure (see [12]). The important distinction between the steady and unsteady case, which allows for a significant simplification of the scheme, is the fact that the parabolic problem can be discretized in time using some of the available highly efficient direction splitting techniques. As we shall see below, this can greatly simplify the computational complexity not only of the coarse-scale problem but also of the local problems and can eventually be generalized to an efficient parallel algorithm for the case when the permeability of the porous media depends significantly on the solution (and therefore on time) and the local subproblems need to be solved often during the course of a simulation. There are many practical situations that yield such problems, most important being the simulation of SAGD (Steam-Assisted Gravity Drainage) oil recovery (see e.g. [13, chapter 9]), and porous media flow in fracking oil and gas recovery. In this paper we propose two techniques for numerical upscaling of parabolic problems with heterogeneous coefficients which achieve a high computational efficiency via the combination of a finite volume spatial discretization and a direction splitting method in time. Additionally, we exploit the structure of the discrete problem at the finest scale so as to devise in a natural way boundary conditions for the local problems which allow to systematically treat problems in which the coefficient varies on the boundaries of the coarse grid cells without using oversampling (overlapping decomposition). The direction splitting decomposes the original problem into a sequence of one-dimensional implicit discrete problems. This in turn allows to compute explicitly the Schur complement matrix corresponding to the interface unknowns corresponding to the boundaries of the coarse grid blocks. This Schur complement is then approximated at the coarse scale by projecting it onto a coarse-scale space defined on the edges of the coarse grid blocks.

The remainder of the paper is organized as follows. In Section 2 we present the direction splitting scheme, and the two possible ways for upscaling of the split and time-discretized system. In Section 3 we present the numerical results using two widely used sets of benchmark data for the coefficient. Finally, we provide some conclusions.

2. Direction splitting multiscale approximation for parabolic equations

2.1. Preliminaries

There is a large variety of schemes that allow for approximating the solution of a parabolic problem by first decomposing the elliptic operator into a sum of positive operators, and then solving a set of parabolic sub problems with each of the operators in this decomposition. For a recent comprehensive review of such methods the reader is referred to the monograph [14]. The main differences between the various schemes are: (i) the order of approximation in time which is usually at most second; (ii) whether the commutativity of the operators in the decomposition is necessary for the unconditional stability of the scheme. Here we consider one of the simplest such schemes that is unconditionally stable for a non-commutative decomposition but has only a first order of accuracy in time. Most of the results can be extended to other available schemes that are second order accurate. Since the stability and convergence results for such schemes, although existing for a long time, are not well known, we summarize them in this section, following the ideas and the presentation in Chapter 5 of [14]. More precisely, we consider the so-called scheme of summarized approximation (in the Western literature, see e.g. [15], this type of schemes are labeled as "Marchuk-Yanenko splitting").

Now we present the abstract setting of the splitting scheme. We assume that H is a Hilbert space with an inner product $(\cdot, \cdot)$ and a corresponding norm and A is a linear operator, which is defined on a dense set in H. We consider the Cauchy problem of finding $\mathsf{u}(t)$ for $t > 0$ such that

$$\begin{cases} \partial_t \mathsf{u} + A\mathsf{u} = f, \quad t > 0, \\ \mathsf{u}(0) = \mathsf{u}^0. \end{cases} \tag{2.1}$$

Here $f(t)$, $t > 0$ and u^0 are given source term and initial data, respectively. The differential operator can supposedly be split into a sum of d operators, with d being the dimension of the problem.

$$A\mathsf{u} := \sum_{\alpha=1}^{d} A^{\alpha}\mathsf{u}. \tag{2.2}$$

For example, one possible operator A and its splitting in 2D, assumed in the remainder of the paper, are given by:

$$A\mathsf{u} := -\partial_x(\lambda(\mathbf{x})\partial_x\mathsf{u}) - \partial_y(\lambda(\mathbf{x})\partial_y\mathsf{u}), \tag{2.3}$$