Computers and Mathematics with Applications 72 (2016) 1655–1669

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

An inverse source problem in a semilinear time-fractional diffusion equation

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ARTICLE INFO

Article history:
Received 12 April 2016
Received in revised form 4 July 2016
Accepted 26 July 2016
Available online 25 August 2016

Keywords:
Semilinear time-fractional diffusion equation
Inverse source problem
Reconstruction
Convergence
Time discretization

ABSTRACT

We study an inverse source problem for a semilinear time-fractional diffusion equation of second order in a bounded domain in $\mathbb{R}^d$. The missing solely time-dependent source is recovered from an additional integral measurement. The existence, uniqueness and regularity of a weak solution is addressed. We design a numerical algorithm based on Rothe's method, derive a priori estimates and prove convergence of iterates towards the exact solution. Theoretical results are supported by a numerical experiment.
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1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary Γ (cf. [1]). Consider a linear second order differential operator in the divergence form with space and time dependent coefficients

$$\begin{aligned} L(x,t)u &= \nabla \cdot (-\boldsymbol{A}(x,t)\nabla u - \boldsymbol{b}(x,t)u) + c(t)u, \\ \boldsymbol{A}(x,t) &= \left(a_{i,j}(x,t)\right)_{i,j=1,\dots,d}, \\ \boldsymbol{b}(x,t) &= (b_1(x,t),\dots,b_d(x,t)). \end{aligned}$$

We deal with a partial differential equation (PDE) with a fractional derivative in time t

$$\left(g_{1-\beta} * \partial_t u(x)\right)(t) + L(x,t)u(x,t) = h(t)f(x) + \int_0^t F(x,s,u(x,s))\,\mathrm{d}s, \quad x \in \Omega,\ t \in (0,T), \tag{1}$$

where $g_{1-\beta}$ denotes the Riemann–Liouville kernel

$$g_{1-\beta}(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}, \quad t > 0,\ 0 < \beta < 1$$

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http://dx.doi.org/10.1016/j.camwa.2016.07.029
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and $*$ stands for the convolution on the positive half-line, i.e.

$$(k * v)(t) = \int_0^t k(t-s)v(s)\,ds.$$

Thus, the convolution term in (1) is the Caputo fractional derivative cf. e.g. [2,3]

$$\partial_t^\beta u(x,t) = \left(g_{1-\beta} * \partial_t u(x)\right)(t).$$

The governing PDE (1) is accompanied by the following initial and boundary conditions

$$\begin{aligned} u(x,0) &= u_0(x), \quad x \in \Omega \\ (-\boldsymbol{A}(x,t)\nabla u(x,t) - \boldsymbol{b}(x,t)u(x,t)) \cdot \nu &= g(x,t) \quad (x,t) \in \Gamma \times (0,T). \end{aligned} \tag{2}$$

The symbol ν denotes the outer normal vector assigned to the boundary Γ.

The integral term in the right-hand-side (r.h.s.) of (1) models memory effects with applications e.g. in elastoplasticity (cf. [4]) or in the theory of reactive contaminant transport [5]. The solvability of forward fractional diffusion equations have been studied e.g. in [6,7]. The *Inverse Source Problem* (ISP) studied in this paper consists of finding a couple $(u(x,t), h(t))$ obeying (1), (2) and

$$\int_\Omega u(x,t)\,dx = m(t), \quad t \in [0,T]. \tag{3}$$

Determination of an unknown source is one of hot topics in inverse problems (IPs). There are many papers studying ISPs in parabolic or hyperbolic settings. If the source exclusively depends on the space variable, one needs an additional space measurement (e.g. solution at the final time), cf. [8–16]. For the solely time-dependent source a supplementary time-dependent measurement is needed, cf. [8,17–19]. This means that both kinds of ISPs need totally different additional data. ISPs for fractional diffusion equations become more popular in the last years. The recovery of a time dependent source in a fractional diffusion equation has been studied in [7,20,21]. Determination of a space dependent function in a fractional diffusion equation has been addressed in [22–25]. The uniqueness of a solution to the inverse Cauchy problem for a fractional differential equation in a Banach space has been studied in [26]. The global existence in time of an ISP for a fractional integrodifferential equation by means of a fixed point method has been considered in [27].

The added value of this paper relies on the global (in time) solvability of the ISP (1), (2), (3), and in the proposition of a very interesting approximation scheme. We reformulate the ISP into an appropriate direct (non-local) formulation. We propose an interesting variational technique based on elimination of h from (1) by (3), which turns out to be possible for a sufficiently smooth solution. Then we prove the well-posedness of the problem. The proposed numerical scheme is based on a semi-discretization in time by Rothe's method cf. [28,29]. We show the existence of approximations at each time step of the time partitioning and we derive suitable stability results. The convergence of approximations towards the exact solution is investigated in Theorem 3.1 in suitable function spaces. Finally, we present a numerical example supporting the obtained convergence results.

2. Uniqueness

Denote by $(\cdot,\cdot)$ the standard inner product of $L^2(\Omega)$ and $\|\cdot\|$ its induced norm. When working at the boundary Γ we use a similar notation, namely $(\cdot,\cdot)_\Gamma$, $L^2(\Gamma)$ and $\|\cdot\|_\Gamma$. If X is a Banach space with the norm $\|\cdot\|_X$, then by $C([0,T],X)$ we denote the set of abstract functions $w : [0,T] \to X$ endowed with the usual norm $\max_{t\in[0,T]} \|\cdot\|_X$. The space $L^p((0,T),X)$ with $p > 1$ is furnished with the norm $\left(\int_0^T \|\cdot\|_X^p \,dt\right)^{\frac{1}{p}}$, cf. [30]. In what follows C, ε and C_ε denote generic positive constants depending only on the given data, where ε is a small one and $C_\varepsilon = C\left(\frac{1}{\varepsilon}\right)$ is a large one.

We associate a bilinear form $\mathscr{L}$ with the differential operator L as follows

$$(Lu, \varphi) = \mathscr{L}(u,\varphi) + (g,\varphi)_\Gamma, \quad \forall \varphi \in H^1(\Omega),$$

i.e.

$$\mathscr{L}(t)(u(t),\varphi) = (\boldsymbol{A}(t)\nabla u(t) + \boldsymbol{b}(t)u(t), \nabla\varphi) + c(t)(u(t),\varphi).$$

Throughout the paper we assume that

$$\begin{aligned} &a_{i,j}, b_i : \overline{\Omega} \times [0,T] \to \mathbb{R}, && |a_{i,j}| + |b_i| \leqslant C, \quad i,j = 1,\dots,d, \\ &0 \leqslant c(t) \leqslant C, && \forall t \in [0,T], \\ &\mathscr{L}(t)(\varphi,\varphi) \geqslant C_0 \|\nabla\varphi\|^2, && \forall \varphi \in H^1(\Omega), \qquad \forall t \in [0,T]. \end{aligned} \tag{4}$$

Integrating (1) over Ω, applying the Green theorem and taking into account (3) we obtain

$$\left(g_{1-\beta} * m'\right)(t) + c(t)m(t) = h(t)(f,1) - (g(t),1)_\Gamma + \int_0^t (F(s,u(s)),1)\,ds. \tag{MP}$$

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