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Multidimensional stability of V-shaped traveling fronts in time periodic bistable reaction–diffusion equations

Wei-Jie Sheng *

Natural Science Research Center, Harbin Institute of Technology, 150080, Harbin, Heilongjiang, People's Republic of China
Aix Marseille Université, CNRS, Centrale Marseille, Institut de Mathématiques de Marseille, UMR 7373, 13453 Marseille, France

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ABSTRACT

This paper deals with the multidimensional stability of time periodic V-shaped traveling fronts in bistable reaction–diffusion equations. It is well known that time periodic V-shaped traveling fronts are asymptotically stable in two dimensional space. In the current study, we further show that such fronts are asymptotically stable under spatially decaying initial perturbations in $\mathbb{R}^n$ with $n \geq 3$. In particular, we show that the fronts are algebraically stable if the initial perturbations belong to L^1 in a certain sense. Furthermore, we prove that there exists a solution oscillating permanently between two time periodic V-shaped traveling fronts, which implies that time periodic V-shaped traveling fronts are not always asymptotically stable under general bounded perturbations. Finally we show that time periodic V-shaped traveling fronts are only time global solutions of the Cauchy problem if the initial perturbations lie between two time periodic V-shaped traveling fronts.

1. Introduction

Traveling fronts have been extensively studied since the seminal works of Fisher [1] and Kolmogorov et al. [2], since they play an essential role in mathematical analysis of evolving spatial patterns generated by evolution equations, see the monograph [3] and the references therein. Recently, a great interest has been drawn to the study of traveling fronts in time heterogeneous environment. See Alikakos et al. [4] for the existence, uniqueness and global stability of time periodic traveling fronts in bistable reaction–diffusion equations; Shen [5,6] for the existence, stability of almost time periodic traveling fronts in bistable reaction–diffusion equations; Zhao and Ruan [7,8], Bao and Wang [9] for the existence and asymptotic stability of time periodic traveling fronts in reaction–diffusion systems. For the study of nonplanar traveling fronts in time heterogeneous environment, one can refer to Wang and Wu [10] for time periodic V-shaped traveling fronts of reaction–diffusion equations with bistable nonlinearity; Sheng et al. [11] for time periodic pyramidal traveling fronts of the same reaction–diffusion equations; Wang [12] for cylindrically symmetric traveling fronts in time periodic bistable reaction–diffusion equation. Other related works on time periodic traveling curved fronts can be referred to [13–19].

It is well known from [3] that stability is an important object in the study of traveling fronts. Recently, the study of the multidimensional stability of traveling fronts has attracted increasing attention. For instance, Xin [20] and Kapitula [21] investigated the stability of planar traveling fronts in multiple space via an application of linear semigroup theory; Levermore and Xin [22] obtained a partial result on the multidimensional stability of bistable reaction–diffusion equation by appealing to the maximum principle and energy methods; Based on the comparison principle and by constructing various supersolutions and subsolutions, Matano et al. [23] and Matano and Nara [24] studied the asymptotic stability

* Correspondence to: Natural Science Research Center, Harbin Institute of Technology, 150080, Harbin, Heilongjiang, People's Republic of China.
E-mail address: shengwj09@hit.edu.cn.

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of planar traveling fronts for the Allen–Cahn equation. Roquejoffre et al. [25] showed that planar traveling fronts are asymptotically stable if the initial perturbations lie between two planar fronts. Very recently, Sheng and Li [26] established the multidimensional stability of time periodic planar traveling fronts of bistable reaction–diffusion equations. Other related works can be referred to [27–30].

Although the multidimensional stability of planar traveling fronts for reaction–diffusion equations has been studied, little attention has been paid to the multidimensional stability of nonplanar traveling fronts. To our knowledge, Sheng et al. [31] established the multidimensional stability of two dimensional V-shaped traveling fronts by employing the comparison principle and supersolutions and subsolutions technique, Chen and Yuan [32] extended the multidimensional stability results of V-shaped traveling fronts [31] to three dimensional pyramidal-shaped traveling front.

However, the multidimensional stability of time periodic V-shaped traveling fronts is still left as an open problem. In the current study we will solve this problem. Actually, we first show that the time periodic V-shaped traveling front is asymptotically stable under spatially decaying initial perturbations. In particular, if the initial perturbations belong to L^1 in a certain sense, we get an algebraic convergence rate which is optimal in some sense. Moreover, we find a solution that oscillates permanently between two time periodic V-shaped traveling fronts. This further indicates that time periodic V-shaped traveling fronts are not asymptotically stable under general bounded perturbations. Finally, we show that time periodic V-shaped traveling fronts are only time global solutions of the Cauchy problem if the initial perturbations lie between two time periodic V-shaped traveling fronts. In this paper, we deal with the multidimensional stability of the following problem

$$u_t = \Delta u + f(u,t), \quad x \in \mathbb{R}^{n-2},\ y \in \mathbb{R},\ z \in \mathbb{R},\ t > 0, \tag{1.1}$$

$$u(x,y,z) = u_0(x,y,z), \quad x \in \mathbb{R}^{n-2},\ y \in \mathbb{R},\ z \in \mathbb{R} \tag{1.2}$$

under the following hypotheses:

(**H1**) $f \in C^{2,1}(\mathbb{R}\times\mathbb{R})$ is T-periodic in t, i.e., there is a $T > 0$ such that $f(\cdot, t+T) = f(\cdot,t)$ for all $t \in \mathbb{R}$.

(**H2**) The period map $P(\alpha) := \omega(\alpha, T)$ has exactly three fixed points $\alpha^-, \alpha^0, \alpha^+$ such that $\alpha^- < \alpha^0 < \alpha^+$, where $\omega(\alpha,t)$ is the solution of

$$\omega_t = f(\omega,t), \quad t \in \mathbb{R}, \qquad \omega(\alpha,0) = \alpha \in \mathbb{R}. \tag{1.3}$$

For simplicity, we write $W^\pm(t) := \omega(\alpha^\pm, t),\ W^0(t) := \omega(\alpha^0,t)$. Furthermore, we assume that there exist constants $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that for all $t \in \mathbb{R}$, there hold

$$W^-(t) + \delta_0 < W^0(t) - \delta_0 < W^0(t) + \delta_0 < W^+(t) - \delta_0,$$
$$\begin{cases} f_\omega(\omega,t) \le -\varepsilon_0 & \text{for } \omega \le W^-(t)+\delta_0 \text{ or } \omega \ge W^+(t)-\delta_0,\\ f_\omega(\omega,t) \ge \varepsilon_0 & \text{for } \omega \in [W^0(t)-\delta_0, W^0(t)+\delta_0] \end{cases} \tag{1.4}$$

and

$$|\omega(t;\omega_0^\pm,\tau) - W^\pm(t)| \to 0 \quad \text{as } t \to \infty, \tag{1.5}$$

where $\omega(t;\omega_0^\pm,\tau)$ is the solution of (1.3) satisfying $\omega(\tau;\omega_0^\pm,\tau) = \omega_0^\pm$ with $\omega_0^- \le \omega(\alpha^0,\tau)$ and $\omega_0^+ \ge \omega(\alpha^0,\tau)$.

A typical example of f satisfying (H1)–(H2) is the cubic potential $f(u,t) = (1-u^2)(2u-\rho(t))$ with $\rho(t) \in (-2,2)$ is T periodic.

Following from Alikakos et al. [4], we know there exist a unique constant s and a unique function $U(\eta,t)$ such that

$$\begin{cases} U_t - U_{\eta\eta} + sU_\eta - f(U,t) = 0, \quad \forall(\eta,t) \in \mathbb{R}^2,\\ U(\pm\infty,t) := \lim\limits_{\eta\to\pm\infty} U(\eta,t) = W^\pm(t), \quad \forall t \in \mathbb{R},\\ U(\cdot,\cdot+T) = U(\cdot,\cdot). \end{cases}$$

In the current study, we are interested in the multidimensional stability of the time periodic V-shaped traveling front of (1.1) in $\mathbb{R}^n$ with $n \ge 3$, where a time periodic V-shaped traveling front is referred to $V(y,\xi,t) = V(y, z+ct, t)$ and $V(\cdot,\cdot,t) = V(\cdot,\cdot,t+T)$ for some positive constant $c > s > 0$. We remark here that the profile equation for V is

$$V_t - V_{yy} - V_{\xi\xi} + cV_\xi - f(V,t) = 0, \quad (y,\xi,t) \in \mathbb{R}^3. \tag{1.6}$$

It is clear that the function V is also a traveling front of (1.1).

Let

$$w(x,y,\xi,t) = u(x,y,z+ct,t), \quad \xi = z+ct.$$

Then $w(x,y,\xi,t)$ satisfies

$$w_t - \Delta w + cw_\xi - f(w,t) = 0, \quad x \in \mathbb{R}^{n-2},\ y \in \mathbb{R},\ \xi \in \mathbb{R},\ t > 0,$$
$$w(x,y,\xi,0) = u_0(x,y,\xi), \quad x \in \mathbb{R}^{n-2},\ y \in \mathbb{R},\ \xi \in \mathbb{R}.$$

For the sake of convenience, we write $w(x,y,\xi,t)$ by $u(x,y,\xi,t)$ and consider the following problem:

$$u_t - \Delta u + cu_\xi - f(u,t) = 0, \quad x \in \mathbb{R}^{n-2},\ y \in \mathbb{R},\ \xi \in \mathbb{R},\ t > 0, \tag{1.7}$$

$$u(x,y,\xi,0) = u_0(x,y,\xi), \quad x \in \mathbb{R}^{n-2},\ y \in \mathbb{R},\ \xi \in \mathbb{R}. \tag{1.8}$$