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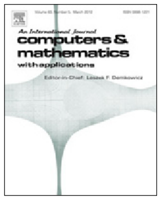

Analytical solutions of multi-term time fractional differential equations and application to unsteady flows of generalized viscoelastic fluid

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A B S T R A C T

This paper derives analytical solutions for a class of new multi-term fractional-order partial differential equations, which include the terms for spatial diffusion, time-fractional diffusion (multi-term) and reaction. These models can be used to describe the nonlinear relationship between the shear stress and shear rate of generalized viscoelastic Oldroyd-B fluid and Burgers fluid. By using a modified separation of variables method, the governing fractional-order partial differential equations are transformed into a set of fractional-order ordinary differential equations. Mikusiński-type operational calculus is then employed to obtain the exact solutions of the linear fractional ordinary differential equations with constant coefficients. The solutions are expressed in terms of multivariate Mittag-Leffler functions. Different situations for the unsteady flows of generalized Oldroyd-B fluid and Burgers fluid due to a moving plate are considered via examples. Integral representations of the solutions are presented. It is shown that the presented results reduce to the corresponding results for classical Navier–Stokes, Oldroyd-B, Maxwell and second-grade fluids as special cases.

1. Introduction

Fractional differential equations have attracted considerable interest in mathematics and many applied fields. Systems of fractional differential equations have been increasingly used to represent physical and control systems, and model many physical and chemical processes in engineering. Recently, fractional differential equations have been proposed for use in modeling dynamical systems of non-Newtonian fluids. Many fluids found in various engineering applications, such as molten plastics, pulps, slurries, emulsions, etc., do not satisfy a linear relationship between the stress tensor and the deformation tensor. These fluids are called non-Newtonian fluids. An important class of non-Newtonian fluids is that of viscoelastic fluids which exhibit both elastic and viscous properties. Among them the Oldroyd-B fluid, which can be used to describe the response of fluids that have slight memory, is widely applied to problems with small relaxation and retardation times and has been intensively studied, see [1–3].

Considerable attention has been devoted to the problem of prediction of the behavior of a non-Newtonian fluid, and rheological constitutive equations with fractional calculus have proved to be a valuable tool to handle viscoelastic properties [4–6]. The fractional models of viscoelastic fluids are derived from the classical equations by replacing the

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integer order time derivative with precise non-integer order integrals or derivatives. A vast literature concerning the flow of generalized Maxwell fluid has been reported, including Stokes' first problem [7], flow on a moving plate [8–10] or between two plates or in a channel [11–13]. As for generalized Oldroyd-B fluid, Tong [14,15] investigated rotational and helical flows of a generalized Oldroyd-B fluid in an annular pipe. Fetecau [16,17] discussed the flow of generalized viscoelastic fluids between two side walls. Hyder [18], Qi [19], Zheng et al. [20–23] and Hayat [24] researched the flow of a generalized Oldroyd-B fluid for different situations. Modified Riemann–Liouville fractional derivatives were used by Jiang and Qi [25] to build a fractional thermal wave model of bioheat transfer. Fan et al. [26] studied the inverse problem for the generalized fractional element network Zener model, and the Bayesian method is introduced to estimate the optimal parameters.

Many numerical methods have been proposed to solve fractional differential equations. Liu et al. [27–40] have proposed some novel numerical methods for fractional differential equations, including fractional method of lines [29], finite difference method [28], finite element method [41,42], finite volume method [43], spectral method [44], meshless method [45]. Qi et al. [10,17,18] and Zheng et al. [9,10,20–23] gave analytical solutions of generalized viscoelastic flows by the Laplace transform method. Luchko [46] considered initial–boundary-value problems for the generalized multi-term time fractional diffusion equation over an open bounded domain.

In this paper we consider analytical solutions for unsteady flows of a generalized Oldroyd-B fluid and a Burgers fluid on a moving plate. A new separation of variables method [38] and modification of Mikusiński's operational calculus [47] for the Caputo fractional derivative are adopted to solve the governing equation. The rest of the paper is organized as follows. Section 2 starts with the derivation of the governing equation. The separation of variables method is adopted to simplify the resulting multi-term time fractional partial differential equations to ordinary differential equations in Section 3. Analytical results are presented in Section 4. We use operational calculus to solve an initial value problem for a general linear fractional differential equation and with the Caputo fractional derivative. Finally, some numerical examples are given in Section 5.

2. Multi-term time fractional dynamical models

The unsteady flow of fluid due to a constantly accelerating plate is considered. The fluid occupies the space $y > 0$ over an infinite plate at $y = 0$. At time $t = 0^+$, the infinite plate begins to slide in its plane with velocity U_0. The fluid near the plate is pulled forward. The conservation equations of mass and momentum are

$$div\,\mathbf{V} = 0, \tag{1}$$

and

$$\rho\frac{d\mathbf{V}}{dt} = -\nabla p + div\,\mathbf{S} + F_b. \tag{2}$$

The constitutive equation for a generalized Oldroyd-B fluid is given by [19,20]

$$\left(1 + \lambda_\alpha \frac{D^\alpha}{\mathrm{D}t^\alpha}\right)\mathbf{S} = \mu\left[1 + \lambda_\beta \frac{D^\beta}{\mathrm{D}t^\beta}\right]\mathbf{A}_1, \quad (0 < \beta \le \alpha < 1). \tag{3}$$

Here $\mathbf{V} = (u, v, w)$ is the fluid velocity, $\mathbf{S} = (S_{ij})$ is the extra-stress tensor, $\mathbf{A}_1 = (\nabla\mathbf{V}) + (\nabla\mathbf{V})^T$ denotes the first Rivlin–Ericksen tensor, ∇ is the gradient operator, p is the pressure, $F_b = (F_{bx}, F_{by}, F_{bz})$ is the body force, ρ is the density of the fluid, μ is the dynamic viscosity coefficient of the fluid. λ_α, λ_β are the material constants, which represent the relaxation time and retardation time, respectively. α and β denote the orders of the fractional derivatives, which are real numbers and satisfy $0 \le \alpha, \beta \le 1$, $D^\alpha/\mathrm{D}t^\alpha$ and $D^\beta/\mathrm{D}t^\beta$ are material derivatives and can be expressed as

$$\frac{D^\alpha\mathbf{S}}{\mathrm{D}t^\alpha} = D_t^\alpha\mathbf{S} + (\mathbf{V}\cdot\nabla)\mathbf{S} - (\nabla\mathbf{V})\mathbf{S} - \mathbf{S}(\nabla\mathbf{V})^T, \tag{4}$$

$$\frac{D^\beta\mathbf{A}_1}{\mathrm{D}t^\beta} = D_t^\beta\mathbf{A}_1 + (\mathbf{V}\cdot\nabla)\mathbf{A}_1 - (\nabla\mathbf{V})\mathbf{A}_1 - \mathbf{A}_1(\nabla\mathbf{V})^T. \tag{5}$$

In previous works [19,20], the Riemann–Liouville fractional-order derivative was considered for the convenience in computation. In this paper, D_t^α and D_t^β are Caputo fractional derivatives of order α and β with respect to t, which are defined as

$$D_t^q f(t) = \frac{1}{\Gamma(m-q)}\int_0^t (t-\tau)^{m-q-1} f^{(m)}(\tau)d\tau, \quad t > 0, \tag{6}$$

in which $m - 1 < q \le m$, $m \in N$, $\Gamma(\cdot)$ is the Gamma function. Note that the model reduces to the classical Oldroyd-B fluid model when $\alpha = \beta = 1$. It reduces to the Maxwell, second grade and Navier–Stokes fluid models for $\lambda_\beta = 0$, $\lambda_\alpha = 0$ and $\lambda_\alpha = \lambda_\beta = 0$ respectively.

The flow is a one-dimensional laminar flow, where the velocity and shear stress take the form

$$\mathbf{V} = u(y, t)\mathbf{i}, \qquad \mathbf{S} = \mathbf{S}(y, t), \tag{7}$$

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