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Error expansion of piecewise constant interpolation rule for certain two-dimensional Cauchy principal value integrals

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ABSTRACT

The classical composite rectangle (midpoint) rule for the computation of two dimensional singular integrals is discussed, with the error functional of rectangle rule for computing two dimensional singular integrals, the local coordinate of certain point and the convergence results $O(h^2)$ are obtained. When the local coordinate is coincided with certain priori known coordinates, we get the convergence rate the same as the Riemann integral at certain point. At last, numerical examples are presented to illustrate our theoretical analysis which agree with it very well.

1. Introduction

We consider the certain two-dimensional Cauchy principal value integral of the form

$$\fint_a^b \fint_c^d \frac{f(x,y)}{(x-t)(y-s)}dxdy = g(t,s), \quad (t,s) \in (a,b) \times (c,d), \tag{1.1}$$

where $\fint_a^b \fint_c^d$ denotes a Cauchy Principle value integral and $(t,s) \in (a,b) \times (c,d)$ the singular point.

Cauchy principal value integrals have recently attracted a lot of attention. The main reason for this interest is probably due to the fact that Cauchy principal value integral equations have shown to be an adequate tool in boundary element methods [1] for the modeling of many physical situations [2–5], such as acoustics, fluid mechanics, elasticity, fracture mechanics and electromagnetic scattering problems and so on. Numerous work have been devoted in developing efficient quadrature formulas, such as the Gaussian method [6], and the Newton–Cote methods [7–15].

In this paper, we focus on certain kind of two-dimensional Cauchy principal value integrals which have been paid less attention to it. In [16] the authors have considered product rules of Gauss type for the numerical approximation of certain two-dimensional Cauchy principal value integrals with respect to generalized smooth Jacobi weight functions. In [17], based on the quadrature of one-dimensional Cauchy principal value integral, generalized quadrature rule for certain two-dimensional Cauchy principal value integrals is presented.

It is the aim of this paper to investigate the convergence phenomenon of rectangle rule for it and, in particular, to derive error estimates. The superconvergence phenomenon of the hypersingular integral is studied in [18–22], and the

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superconvergence phenomenon of the Cauchy principal value integral is studied in [23–27]. In this paper, the classical rectangle rule for computation of certain kind of two-dimensional Cauchy principal value integrals is presented, then the error functional of numerical rule is given and with the special function equals zero, the convergence phenomenon is obtained when the local coordinate coincides with zero point of special function. This can be considered as the above generalized one-dimensional convergence results to cover this new situation. Moreover, we give an easy estimate of the corresponding remainder when the density function $f(x, y)$ belongs to C^3.

The rest of this paper is organized as follows. In Section 2, after introducing some basic formulas of the classical rectangle rule, we present the main results. In Section 3 we finished the proof of main results. Finally, several numerical examples are provided to validate our analysis.

2. Main result

Let $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ and $c = y_0 < y_1 < \cdots < y_{m-1} < y_m = d$ be a uniform partition of the area $[a, b] \times [c, d]$ with mesh size $h_x = (b-a)/n$ and $h_y = (d-c)/m$. In order to simplify our analysis, we set $h = h_x = h_y = (b-a)/n = (d-c)/m$, it is not difficult to extend our analysis to the quasi-uniform meshes.

Define $f_C(\hat{x}_i, \hat{y}_j)$ as the piecewise constant interpolant for $f(x, y)$:

$$f_C(\hat{x}_i, \hat{y}_j) = f(\hat{x}_i, \hat{y}_j), \qquad \hat{x}_i = \frac{x_i + x_{i+1}}{2}, \qquad \hat{y}_j = \frac{y_j + y_{j+1}}{2}, \quad i, j = 0, 1, \ldots, n-1 \tag{2.1}$$

where

$$f_C(\hat{x}_i, \hat{y}_j) = f(x, y) + f_x(x, y)(\hat{x}_i - x) + f_y(x, y)(\hat{y}_j - y) + R(\alpha_{1i}, \beta_{1j}), \quad i, j = 0, 1, \ldots, n-1$$

here

$$R(\alpha_{1i}, \beta_{1j}) = \frac{1}{2} f_{xx}(\alpha_{1i}, \beta_{1j})(\hat{x}_i - x)^2 + f_{xy}(\alpha_{1i}, \beta_{1j})(\hat{x}_i - x)(\hat{y}_j - y) + \frac{1}{2} f_{yy}(\alpha_{1i}, \beta_{1j})(\hat{y}_j - y)^2$$

with

$$\alpha_{1i} = \begin{cases} (\hat{x}_i, x) & x \geq \hat{x}_i \\ (x, \hat{x}_i) & x \leq \hat{x}_i, \end{cases} \qquad \beta_{1j} = \begin{cases} (\hat{y}_j, y) & y \geq \hat{y}_j \\ (y, \hat{y}_i) & y \leq \hat{y}_j. \end{cases}$$

Then we get

$$\begin{aligned} I(f, t, s) &:= \int_a^b \int_c^d \frac{f_C(\hat{x}_i, \hat{y}_j)}{(x-t)(y-s)} dxdy \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} f(\hat{x}_i, \hat{y}_j) \omega_{i,j}(t, s) + E_n(f, t, s) \\ &= I_n(f, t, s) \end{aligned} \tag{2.2}$$

where $E_n(f, t, s)$ denotes the error functional and

$$\omega_{i,j}(t, s) = \frac{h_x}{\hat{x}_i - t} \frac{h_y}{\hat{y}_j - s}. \tag{2.3}$$

To define the constant interpolation polynomials and a linear transformation

$$x = \hat{x}_i(\tau) := (\tau + 1)(x_{i+1} - x_i)/2 + x_i, \quad \tau \in [-1, 1],$$
$$y = \hat{y}_j(\xi) := (\xi + 1)(y_{j+1} - y_j)/2 + y_j, \quad \xi \in [-1, 1]$$

from the reference element $[-1, 1]$ to the subinterval $[x_i, x_{i+1}]$ and $[y_j, y_{j+1}]$. Here we present the error estimate for the (composite) rectangle rule with certain Two-Dimensional Cauchy principal value integrals in the following theorem.

Theorem 2.1. *Assume $f(x, y) \in C^1[a, b] \times [c, d]$. For the rectangle rule $I_n(f, t, s)$ defined as (2.2). Assume that $t = x_k + (1 + \tau)h/2$, $\tau \in [-1, 1]$, $\tau \neq 0$, $s = y_l + (1 + \xi)h/2$, $\xi \in [-1, 1]$, $\xi \neq 0$, there exists a positive constant C, independent of h and s, such that*

$$|E_n(f, t, s)| \leq C[|\ln h|^2 + \gamma^{-1}(\tau) + |\ln h|(\eta(t) + \eta(s))]h, \tag{2.4}$$

where

$$\gamma(\tau) = \gamma(\xi) = \min_{0 \leq i \leq n-1} \frac{|t - x_i|}{h} = \min_{0 \leq j \leq n-1} \frac{|s - y_j|}{h} = \frac{1 - |\tau|}{2} \tag{2.5}$$

and

$$\eta(t) = \frac{1}{(t-a)(b-t)}, \qquad \eta(s) = \frac{1}{(s-c)(d-s)}. \tag{2.6}$$

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