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On high order methods for the heterogeneous Helmholtz equation

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ABSTRACT

The heterogeneous Helmholtz equation is used in geophysics to model the propagation of a time harmonic wave through the earth. Processing seismic data (inversion, migration...) involves many solutions of the Helmholtz equation, so that an efficient numerical algorithm is required. It turns out that numerical approximation of waves becomes very demanding at high frequencies because of the pollution effect. In the case of homogeneous media high order methods can reduce the pollution effect significantly, enabling the approximation of high frequency waves. However, they fail to handle fine-scale heterogeneities and cannot be applied as-is to heterogeneous media. In this paper, we show that if the propagation medium is properly approximated using a multiscale strategy, high order methods are able to capture subcell variations of the medium. Furthermore, focusing on a one dimensional model problem enables us to prove frequency explicit asymptotic error estimates, showing the superiority of high order methods. Numerical experiments validate our approach and comfort our theoretical results.

0. Introduction

Numerical approximation of high frequency waves is a challenging problem: because of the pollution effect, the mesh needs to be drastically refined. In the context of Galerkin methods the pollution effect is the fact that if the number of degrees of freedom per wavelength is fixed, the error of the best approximation of the discrete space remains bounded, while the numerical solution is diverging from the true solution when the frequency is increasing. This is due to the fact that the Helmholtz operator is not coercive, so that quasi-optimality of the discrete scheme is not ensured for arbitrary meshes.

The pollution effect has been extensively studied in the case of homogeneous media. In particular, it is known that even if it is possible to design pollution-free schemes in one dimension, it is not the case in two and three dimensions as shown by Babuška and Sauter in [1].

Frequency explicit error estimates have been derived: the finite element error is bounded explicitly in terms of the frequency ω, the mesh step h and the order of discretization p. Two types of results are available. First, if the mesh is fine enough, the finite element solution is quasi-optimal. This results are called asymptotic error estimates. The second type of results are the so-called pre-asymptotic error-estimates. They give an optimal condition on the mesh to bound the error independently of the frequency.

For the case of one dimensional homogeneous media, optimal pre-asymptotic error-estimates for Lagrangian polynomial discretizations have been established by Babuška and Ihlenburg in the pair of papers [2,3]. It is shown that the error can be

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decomposed into two different factors. The best approximation error, of order $\omega^p h^p$, and the phase lag, of order $\omega^{2p+1}h^{2p}$. In particular, the error is bounded independently of the frequency, if $h \simeq \omega^{-1-1/(2p)}$. The pollution effect is present for all p, since the mesh step must satisfy $h \simeq \omega^{-1-1/(2p)} \ll \omega^{-1}$. However, this effect is reduced for high order methods, since the exponent on ω gets closer to one as p is increasing.

Asymptotic error estimates are available in two and three dimensions. Melenk and Sauter showed in [4,5] that finite element schemes are stable and that the finite element solution is quasi-optimal under the condition that $\omega^{p+1}h^p < C$.

In this paper, we focus on the case of highly heterogeneous media. Classical high order discretizations fail to handle such propagation media, because they are not able to see the fine scales of the velocity parameter. Indeed, they are build upon coarser meshes than low order methods. Therefore, if the velocity parameter is taken to be constant in each cell (through averaging, or local homogenization strategies), fine scale information is (at least partially) lost. Furthermore, restricting the mesh steps so that the velocity parameter is constant inside each cell usually higher the computational cost too much if the medium is highly heterogeneous.

We propose to overcome this difficulty with a Multiscale Medium Approximation method (MMAm). The velocity parameter is not assumed to be constant on each cell, but on a submesh of each cell. If the submeshes are designed properly, the MMAm is equivalent to a quadrature formula, adapted to the medium. In particular, we show that this methodology has roughly the same computational cost as the classical finite element method. The method was presented in [6] for a two-dimensional Helmholtz problem and a pre-asymptotic error estimate has been demonstrated for linear elements.

The aim of this paper is to extend the analysis of the MMAm to higher order discretizations. Though practical applications are 3D, we focus on a one dimensional model problem. This choice enables us to simplify the proofs, but most of our results are extensible to higher dimensions.

First, we show that the heterogeneous Helmholtz problem is well-posed and derive frequency-explicit stability estimates with respect to the right hand side, and with respect to variations of the velocity parameter, justifying the use of medium approximation. Those results are obtained assuming the velocity parameter is monotonous and that the propagation medium is surrounded by first order absorbing boundary conditions. However, these hypotheses are not mandatory to discretize the problem.

Second, we derive asymptotic error estimates for the MMAm. Even if the solution can be rough inside each cell because of velocity jumps, we are able to extend the asymptotic error estimates obtained in [4] for $1 \leq p \leq 3$.

Third, we investigate numerically the stability of the scheme when the frequency is increasing to figure out optimal meshing conditions. We show that in simple media, the optimal homogeneous pre-asymptotic error estimates is still valid. However, in more complex cases, it looks like the condition $h \simeq \omega^{-1-1/(2p)}$ is not sufficient anymore.

Finally, we fix the frequency and compare different orders of discretization to achieve a given precision. We are able to conclude that high order methods are interesting: in our examples, $p = 4$ discretizations always yield a smaller linear system than lower order discretizations for the same precision.

The paper is organized as follows: in Section 1, we define our model problem and introduce the notations we will be using in the remaining. Section 2 is devoted to the analysis of the continuous problem. We analyse the MMAm in Sections 3–5 and numerical experiments are presented in Sections 6 and 7.

1. Settings

We consider the Helmholtz equation set in the heterogeneous one dimensional domain $(0, Z)$ with absorbing boundary conditions

$$\begin{cases} -\dfrac{\omega^2}{c^2(z)}u(z) - u''(z) &= f(z), \quad z \in (0, Z), \\ -u'(0) - \dfrac{\mathbf{i}\omega}{c(0)}u(0) &= 0, \\ u'(Z) - \dfrac{\mathbf{i}\omega}{c(Z)}u(Z) &= 0, \end{cases} \tag{1}$$

where f is the load term, ω is the pulsation and c is the velocity parameter. Since we especially focus on the high frequency case, we will consider real frequencies $\omega \geq 1$.

For the sake of simplicity, we restrict ourself to the case where c is piecewise constant. We could also have considered piecewise smooth parameters. We do not because we consider that the most difficult part of the analysis is the jumps of the parameter, which are considered herein. We will also assume that all the parameters we consider are uniformly bounded above and below by two constants. This is a reasonable assumption, which can be justified in geophysics by the properties of rocks. We also introduce two additional hypothesis which are required for our theoretical analysis. We will assume that the length of the thinnest layer is bounded below and that the velocity parameter is monotonous. Remark that the monotonous hypothesis is still valid in a lot of geophysical application, since the wave velocity is usually increasing with depth. Our assumptions on c are summarized in Definition 1.

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