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Instability in a generalized multi-species Keller–Segel chemotaxis model

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ABSTRACT

In this paper we discuss Turing instability of a generalized Keller–Segel chemotaxis model involving n-species and m-chemoattractants. By using combinatorial matrix theory we extend the result on the instability of a single-species and n-chemicals Keller–Segel system (Leenheer et al., 2012) to a general $(n+m)\times(n+m)$ system. A sufficient condition for chemotaxis driven Turing instability of the multi-species model is obtained. Moreover, some examples and numerical simulations are presented to illustrate the theoretical results.

1. Introduction

A large amount of research has been devoted to the study of symmetry breaking instabilities leading to steady-state solutions in models for chemical and biological pattern formation employing reaction–diffusion systems. In 1952, Alan Turing [1] showed that the combined effect of diffusion and chemical reaction may result in destabilizing a uniform equilibrium. He suggested that, under certain conditions, chemical can react and diffuse in such a way as to produce nonconstant equilibrium solution, which represents spatial patterns of chemical or morphogen concentration.

The majority of theoretical studies in reaction–diffusion theory focus on the analysis of systems composed of only two species (or one species and one chemical) with kinetics chosen to interact in the way necessary for a Turing instability. While this approach captures the essence of the Turing instability, it is often not realistic since chemical and biochemical reactions usually involve more than two dynamically independent species [2].

In [3], the authors considered semilinear parabolic systems with three interacting species. They showed that a necessary condition for a Turing instability to take place in these systems is that they must contain an unstable subsystem, which may be composed of one or two species.

For general n-dimensional semilinear reaction–diffusion systems, Satnoianu, Menzinger and Maini [2] presented necessary and sufficient conditions on the stability matrix which guarantee that its uniform steady state can undergo a Turing bifurcation. The necessary condition, requiring that the system be composed of an unstable and a stable subsystem, and the sufficient condition of sufficiently rapid inhibitor diffusion relative to the activator subsystem are established.

In the recent paper [4] the authors considered Turing instability of a n-species quasilinear reaction–diffusion system with density-dependent diffusion in a d-dimensional box $(0,\pi)^d (d=1,2,3)$. A sufficient condition of linear instability for the system is derived. In particular, they proved that the unstable equilibrium point of the system is nonlinearly unstable using a bootstrap technique, which was first introduced in [5].

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Chemotaxis is the capacity of organisms to move along a chemical gradient. Such movement, maybe towards or away from a higher concentration of the chemical substance, has been investigated by different authors, not only from a biological point of view but also from mathematical, physical, or chemical perspectives, among others. In particular, in the early 1970s, Keller and Segel [6] proposed a quasilinear system of two parabolic equations to describe the aggregation of Dictyostelium discoideum, a soil-living amoeba.

In [7], author extended the Keller–Segel model to some multi-species chemotaxis equations. Several natural questions that have attracted the attention of several researchers in the context of the Keller–Segel model are addressed in [7] for the multi-species and multi-chemicals models. They include blow-up, global existence of solutions, existence and non-existence of non-uniform steady states, destabilization of uniform steady states as a way to achieve pattern formation, and the existence of Lyapunov functionals. Although [7] begins with this very general model, the focus of the article quickly turns to specific cases. In [7, Section 3], a linear stability analysis is performed for uniform steady states of a single-species, two chemical systems. By the main results in the above Refs. [2–4,7] we notice that the Routh–Hurwitz criterion plays a key role in linearized stability analysis for multi-species reaction–diffusion systems. The problem of detecting Diffusion Driven Instability (DDI) is key to predicting the onset of pattern formation in spatially distributed biological and ecological systems. However, detecting DDI by standard techniques based on eigenvalues is often difficult, especially for large systems.

In [8] Leenheer et al. presented the following single-species, n-chemicals Keller–Segel model without growth

$$\begin{cases} u_t = \nabla(D\nabla u - \chi u \nabla v), & x \in \Omega, t > 0, \\ v_t = A\Delta v + \alpha u + g(v), & x \in \Omega, t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases}$$

where $u(x,t)$ denotes the density of species, $v_j(x,t)$ represents the concentration of jth chemical attractant at a position x and a time t, $v(x,t) = (v_1(x,t), v_2(x,t) \cdots v_n(x,t))$. The positive constants D, χ are the random diffusion rate and the chemotactic sensitivity, respectively. The matrix A =diag$(A_1, A_2 \cdots A_n)$, $A_i > 0, i = 1, 2 \cdots n$. By using matrix theoretic tools [9], Leenheer et al. provided easily verifiable sufficient conditions for destabilizing the homogeneous steady states.

Very recently, Negreanu and Tello [10] studied the stability of homogeneous steady states of a two-species, one slow diffusive chemical Keller–Segel model without growth. They in [11] also considered the asymptotic stability of the two species chemotaxis system with non-diffusive chemoattractant and logistic growth by using an iterative approach and energy method. Tello and Winkler [12] studied the global asymptotic stability of a two-species parabolic–parabolic–elliptic chemotaxis system with a logistic source by using a comparative technique. Wang and Li [13] extended the result in [12] to a chemotaxis system involving more than two competitive species all of which are attracted by the same chemoattractant.

In this short paper, we consider the Turing instability of the following multi-species Keller–Segel chemotaxis model

$$\begin{cases} u_t = \nabla(D(u,v)\nabla u - C(u,v)\nabla v) + H(u,v), & x \in \Omega,\ t > 0, \\ v_t = \nabla(A(u,v)\nabla v) + G(u,v), & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $R^N (N \geq 1)$, ν denotes the outward unit normal on the domain boundary $\partial\Omega$. $u_i(x,t)$ denotes the density of ith specie at a position x and a time t, $u(x,t) = (u_1(x,t), u_2(x,t), \ldots, u_n(x,t))$, and $v_j(x,t)$ represents the concentration of jth chemoattractant, $v(x,t) = (v_1(x,t), v_2(x,t), \ldots, v_m(x,t))$. The matrices $D(u,v) = \text{diag}(D_1(u,v), D_2(u,v), \ldots, D_n(u,v))$, $A(u,v) = \text{diag}(A_1(u,v), \ldots, A_m(u,v))$, $C(u,v) = (C_{ij}(u,v))_{n\times m}$. $H(u,v) = (H_1(u,v), H_2(u,v), \ldots, H_n(u,v))$, and $G(u,v) = (G_1(u,v), G_2(u,v), \ldots, G_m(u,v))$. Throughout the article we always assume that $D(u,v)$, $A(u,v)$, $C(u,v)$, $H(u,v)$, $G(u,v) \in C^2$.

In the following section we shall extend to the multi-species multi-chemicals Keller–Segel chemotaxis model with growth, the instability result on the homogeneous steady state of the $1 + n$ Keller–Segel model without growth in [8] by using the combinatorial matrix theory.

2. Instability of the homogeneous steady state

In this section, we first review some basic facts concerning the notations and terminology from the combinatorial matrix theory [9] used throughout this paper. Then we will focus on the Turing instability in the Keller–Segel chemotaxis model (1.1).

2.1. Some auxiliary results

Definition 2.1. A square matrix is called a Metzler matrix if its off-diagonal entries are non-negative.

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