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# On regularity criteria for the 3D Navier–Stokes equations involving the ratio of the vorticity and the velocity

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ABSTRACT

This note concerns regularity criteria for the Navier–Stokes equations. It is proved that if the solution satisfies

$$\int_0^T \frac{\|\boldsymbol{\omega}(\tau)\|_{L^s}^{\frac{2s}{2s-3}}}{\|\boldsymbol{u}(\tau)\|_{L^3}^{f(s)}}\,\mathrm{d}\tau < \infty$$

for $\frac{3}{2} < s < \infty$ and suitable function $f(s)$, then the solution is regular on $(0, T]$.


## 1. Introduction

In this paper, we are concerned with regularity criteria for the weak solutions to the incompressible Navier–Stokes equations (NSE) in $\mathbb{R}^3$:

$$\begin{cases}\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \triangle\boldsymbol{u} + \nabla\pi = \boldsymbol{0},\\ \nabla\cdot\boldsymbol{u} = 0,\\ \boldsymbol{u}(0) = \boldsymbol{u}_0,\end{cases} \tag{1.1}$$

where $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\pi$ denote the unknown velocity field and pressure of the fluid respectively, and $\boldsymbol{u}_0$ is the prescribed initial data satisfying the compatibility condition $\nabla\cdot\boldsymbol{u}_0 = 0$. Here and in what follows, we shall use the notations:

$$\partial_t\boldsymbol{u} = \frac{\partial\boldsymbol{u}}{\partial t}, \qquad \partial_i = \frac{\partial}{\partial x_i}, \qquad (\boldsymbol{u}\cdot\nabla) = \sum_{i=1}^3 u_i\partial_i.$$

The existence of a weak solution

$$\boldsymbol{u} \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)) \tag{1.2}$$

of (1.1) has been established in the pioneer works of Leray [1] and Hopf [2] (for the case of bounded domains). However, the issue of regularity and uniqueness of such a weak solution remain open up to now. The classical Prodi–Serrin conditions

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(see [3–5]) say that if

$$\boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \qquad \frac{2}{p}+\frac{3}{q}=1, \quad 3 \leqslant q \leqslant \infty, \tag{1.3}$$

then the solution is smooth on $(0,T)$. The difficult limiting case of (1.3) was recently treated in [3] by using backward uniqueness method.

From the scaling point of view, (1.3) is important in the sense that

$$\|\boldsymbol{u}_\lambda\|_{L^p(0,t;L^q(\mathbb{R}^3))} = \|\boldsymbol{u}\|_{L^p(0,\lambda T;L^q(\mathbb{R}^3))} \tag{1.4}$$

for

$$\frac{2}{p}+\frac{3}{q}=1, \quad \text{with } \boldsymbol{u}_\lambda(x,t)=\lambda\boldsymbol{u}(\lambda x,\lambda^2 t)\ (\lambda>0).$$

Regularity criterion (1.3) was later generalized by Beirão da Veiga [6] to be

$$\nabla\boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \qquad \frac{2}{p}+\frac{3}{q}=2, \quad \frac{3}{2} \leqslant q \leqslant \infty. \tag{1.5}$$

Recently, Tran–Yu [7] discovered the pressure driving the Navier–Stokes flows has a high degree of nonlinear depletion, and established a series of regularity criteria involving the ratio of physically quantities. Then Tran [8] studied the ratio of the vorticity and the velocity in a moderate and effective way.

One among others, Tran–Yu [7, Corollary 2] established the regularity condition

$$\int_0^T \frac{\|\boldsymbol{\omega}(\tau)\|_{L^2}^4}{\|\boldsymbol{u}(\tau)\|_{L^3}^2}\,\mathrm{d}\tau < \infty, \tag{1.6}$$

which was generalized in [8, Theorem 1] to be

$$\int_0^T \frac{\|\boldsymbol{\omega}(\tau)\|_{L^2}^4}{\|\boldsymbol{u}(\tau)\|_{L^3}^3}\,\mathrm{d}\tau < \infty, \tag{1.7}$$

where $\boldsymbol{\omega}=\nabla\times\boldsymbol{u}$ is the vorticity. These results are very interesting. Firstly, in the numerator, the vorticity is in the Serrin's class. Secondly, in the denominator, the velocity has its critical norm.

The aim of this note is to extend (1.7) to general Prodi–Serrin type regularity criteria.

**Theorem 1.1.** *Let $\boldsymbol{u}_0 \in L^3(\mathbb{R}^3)$. Assume that $\boldsymbol{u}$ is a weak solution of (1.1) in the sense of Leray and Hopf. If one of the following three conditions holds,*

$$\int_0^T \frac{\|\boldsymbol{\omega}(\tau)\|_{L^s}^{\frac{2s}{2s-3}}}{\|\boldsymbol{u}(\tau)\|_{L^3}^3}\,\mathrm{d}\tau < \infty, \quad \frac{3}{2} < s \leqslant \frac{15}{8}; \tag{1.8}$$

$$\int_0^T \frac{\|\boldsymbol{\omega}(\tau)\|_{L^s}^{\frac{2s}{2s-3}}}{\|\boldsymbol{u}(\tau)\|_{L^3}^{\frac{2(3-s)}{2s-3}}}\,\mathrm{d}\tau < \infty, \quad \frac{15}{8} < s < 2; \tag{1.9}$$

$$\int_0^T \frac{\|\boldsymbol{\omega}(\tau)\|_{L^s}^{\frac{2s}{2s-3}}}{\|\boldsymbol{u}(\tau)\|_{L^3}^{\frac{3}{2s-3}}}\,\mathrm{d}\tau < \infty, \quad 2 \leqslant s < \infty, \tag{1.10}$$

*then the solution is smooth on* $(0,T]$.

Before proving Theorem 1.1 in Section 2, let us first recall some inequalities we shall often use. Taking the divergence of $(1.1)_1$ gives

$$-\Delta p = \sum_{i,j=1}^{3} \partial_i\partial_j(u_iu_j),$$

and thus by classical elliptic estimates,

$$\|p\|_{L^q} \leqslant C\,\|\boldsymbol{u}\|_{L^{2q}}^2, \quad 1<q<\infty. \tag{1.11}$$

By Hölder inequality, we have further

$$\|p|\boldsymbol{u}|^\alpha\|_{L^q} \leqslant \|p\|_{L^{\frac{(2+\alpha)q}{2}}}\,\|\boldsymbol{u}\|_{L^{(2+\alpha)q}}^\alpha \leqslant C\,\|\boldsymbol{u}\|_{L^{(2+\alpha)q}}^{2+\alpha}, \quad 1<q<\infty,\ \alpha \geqslant 0. \tag{1.12}$$

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