Computers and Mathematics with Applications 72 (2016) 2315–2333

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Solving two dimensional second order elliptic equations in exterior domains using the inverted finite elements method

Samir Kumar Bhowmik[a,*], Rabah Belbaki[b], Tahar Zamene Boulmezaoud[c], Samy Mziou[d]

[a] Department of Mathematics, University of Dhaka, Dhaka 1000, Bangladesh

[b] Ecole Normale Supérieure de Kouba, Department of Mathematics, Algiers, Algeria

[c] Laboratoire de Mathématiques de Versailles, Université de Versailles Saint-Quentin-en-Yvelines, 45, avenue des Etats-Unis, 78035, Versailles, cedex, France

[d] IMSIU, College of Sciences, Department of Mathematics and Statistics, P.O. Box 90950, Riyadh 11623, Saudi Arabia

ARTICLE INFO

Article history:
Received 27 January 2016
Received in revised form 11 August 2016
Accepted 28 August 2016
Available online 17 October 2016

Keywords:
Inverted finite elements
Exterior domain
Elliptic equation
Unbounded domains

ABSTRACT

In this paper, inverted finite element method is used for solving two-dimensional second order elliptic equations with a Dirichlet boundary condition in an exterior domain. After laying down the method, and after giving an estimate of the error, we detail how its implementation can be accomplished. Numerical results show the high efficiency and the accuracy of the method, especially for equations with infinitely varying coefficients.

1. Introduction

Several models in physics and in engineering lead to two dimensional elliptic partial differential equations outside a bounded obstacle. These equations can be written as

$$-\text{div}\,(a\nabla u) + \boldsymbol{b}.\nabla u + cu = f, \quad \text{in } \Omega, \tag{1.1}$$

where Ω is an exterior domain, that is

$$\Omega = \mathbb{R}^2 \setminus \overline{\omega},$$

with $\omega \subset \mathbb{R}^2$ a bounded open set. Here a, $\boldsymbol{b}$ and c are coefficients which could vary at large distances.

In this paper, focus is on issues emerging from finding approximate solution to Eq. (1.1) completed with the Dirichlet boundary condition

$$u = u_0 \quad \text{on } \partial\Omega, \tag{1.2}$$

and an appropriate asymptotic condition when $|x| \to +\infty$. Addressing such issues requires to take into account the far-field behavior of the solution and the complications due to the unboundedness of the geometric domain. Actually, the

* Corresponding author.
E-mail addresses: bhowmiksk@gmail.com, bhowmiksk@du.ac.bd (S.K. Bhowmik), belbakirab@gmail.com (R. Belbaki), tahar.boulmezaoud@uvsq.fr (T.Z. Boulmezaoud), mzious@gmail.com (S. Mziou).

http://dx.doi.org/10.1016/j.camwa.2016.08.030
0898-1221/

performance of numerical methods strongly depend on the treatment of the behavior of the searched solution at large distances.

First of all, let us recall that most of the existing numerical methods are based on a truncation of the computational domain at some distance or, whenever possible, on rewriting the original problem as a boundary integral equation. Among these methods, we can quote Boundary element methods (BEM) (see, e.g., Brebbia et al. [1], Ciskowski and Brebbia [2], Beer et al. [3] and Colton and Kress [4]), Artificial boundary condition methods (see, e.g., [5–9]), Absorbing Boundary Conditions (ABC) methods (see, e.g., Enquist and Majda [10,11], or Bayliss and Turkel [12]) or Perfectly Matched Layers (PML) methods (see Bérenger [13]). Among non truncature methods we can mention the Infinite Element Method (IEM) (see Bettess [14], Bettess and Zienkiewicz [15], Burnett [16], Gerdes and Demkowicz [17], Shirron and Babuvska [18] and Toselli [19]) or spectral-like methods (see Shen and Wang [20] or Boulmezaoud et al. [21]). However, we may note that some of these mentioned investigations deal with wave-like solutions and can be easily adapted to problems without oscillations. Secondly, most of these methods deal with elliptic partial differential equations whose coefficients do not vary (or approach constants) at large distances.

Here, our main objective is to use an Inverted Finite Element Method (IFEM) for approximating solutions of (1.1) completed with a Dirichlet's boundary condition (1.2) and a prescribed behavior in far reaches of Ω, that is when $|x| \to +\infty$. More precisely, we require that u comply in some ways with the following condition

$$u(x) = \circ((\log|x|)^{1/2}) \quad \text{when } |x| \to +\infty \tag{1.3}$$

or with the more restrictive condition

$$u(x) = \circ(|x|^{-\alpha}) \quad \text{when } |x| \to +\infty \tag{1.4}$$

for some $\alpha \geq 0$. In doing so, we are not only considering equations with constant coefficients, but also those having indefinitely varying coefficients a, b and c. Notice that IFEM was originally proposed by Boulmezaoud in [22] (see also, e.g., [23,24]). The method preserves unboundedness of the domain and its implementation closely resembles that of the finite element method, except the special treatment of boundlessness of the domain by means of polygonal inversions. We may note that in this original paper, the author laid down the principles of the method disregarding the treated partial differential equation, which is supposed to be formulated in an adequate functional framework. Details about this formulation are not given since they depend on the treated equation and on the nature of the geometrical domain which could be, e.g., the whole space, an exterior domain or the half-space. In addition, this underlying functional framework uses a family of non logarithmic weighted Sobolev spaces which is appropriated for solving N-dimensional second order elliptic problems with $N \neq 2$. Our main innovation here is to show that IFEM can be tailored for solving external second order equations of the form (1.1) when $N = 2$, provided that the problem is correctly settled in a logarithmic weighted space. We would note that we do not consider here problems with Neumann or Robin boundary conditions, although their treatment is quite similar. The main reason for this is that addressing such problems requires some non-negligible technical modifications (which will be considered in a short forthcoming paper).

The remaining of this paper is organized as follows. In Section 2, we disclose the underlying functional framework (logarithmic weighted function spaces). Then, we recast the problem in a variational form and we prove its well posedness. In Section 3, we shall show in a well-rounded and detailed manner how IFEM can be used successfully for solving this problem defined in the exterior of an obstacle. We also prove the convergence of the method and we discuss the implementation in detail. Lastly, in Section 4, we exhibit some computational results which confirm the performance of the method.

2. The underlying spaces. Variational formulation

One of the objectives of this section is to introduce some functional tools which are useful to properly formulate the problem. The purpose is not only to analyze the continuous problem, but also to prepare an appropriate framework for developing the IFEM method in next sections.

In the sequel, ω denotes a bounded open and connected subset of $\mathbb{R}^2$ having a Lipschitzian boundary. We set $\Omega = \mathbb{R}^2 \setminus \overline{\omega}$ and we consider the space $W^1_{\log}(\Omega)$ composed of all generalized functions v satisfying

$$\int_\Omega \frac{|v|^2}{(|x|^2+1)(\log(|x|^2+2))^2}dx < +\infty, \qquad \int_\Omega |\nabla v|^2 dx < \infty, \tag{2.1}$$

where $|x| = (x_1^2 + x_2^2)^{1/2}$ for $x = (x_1, x_2) \in \Omega$ and endowed with the norm

$$\|v\|^2_{W^1_{\log}(\Omega)} = \int_\Omega \frac{|v|^2}{(|x|^2+1)(\log(|x|^2+2))^2}dx + \int_\Omega |\nabla v|^2 dx.$$

We set

$$\mathring{W}^1_{\log}(\Omega) = \{v \in W^1_{\log}(\Omega) \mid v = 0 \text{ on } \partial\omega\}.$$