ARTICLE IN PRESS

Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A numerical study of compact approximations based on flat integrated radial basis functions for second-order differential equations

C.M.T. Tien, N. Mai-Duy, C.-D. Tran, T. Tran-Cong *

Computational Engineering and Science Research Centre, Faculty of Health, Engineering and Sciences, The University of Southern Queensland, 487/521-535 West Street, Toowoomba, Queensland 4350, Australia

ARTICLE INFO

Article history:
Received 1 September 2015
Received in revised form 26 August 2016
Accepted 1 September 2016
Available online xxxx

Keywords:
RBF
Integrated RBF
Shape parameter
Ill-conditioning
ODE
PDE

ABSTRACT

In this paper, we propose a simple but effective preconditioning technique to improve the numerical stability of Integrated Radial Basis Function (IRBF) methods. The proposed preconditioner is simply the inverse of a well-conditioned matrix that is constructed using non-flat IRBFs. Much larger values of the free shape parameter of IRBFs can thus be employed and better accuracy for smooth solution problems can be achieved. Furthermore, to improve the accuracy of local IRBF methods, we propose a new stencil, namely Combined Compact IRBF (CCIRBF), in which (i) the starting point is the fourth-order derivative; and (ii) nodal values of first- and second-order derivatives at side nodes of the stencil are included in the computation of first- and second-order derivatives at the middle node in a natural way. The proposed stencil can be employed in uniform/nonuniform Cartesian grids. The preconditioning technique in combination with the CCIRBF scheme employed with large values of the shape parameter are tested with elliptic equations and then applied to simulate several fluid flow problems governed by Poisson, Burgers, convection–diffusion, and Navier–Stokes equations. Highly accurate and stable solutions are obtained. In some cases, the preconditioned schemes are shown to be several orders of magnitude more accurate than those without preconditioning.

1. Introduction

During the last three decades, Radial Basis Function (RBFs) have found increasingly widespread use for numerical solution to the Partial Differential Equation (PDE) systems. Hardy [1,2] devised Multi-Quadric (MQ) RBF schemes for scattered data fitting and general multi-dimensional data interpolation problems in geo-physical engineering. Buhmann [3] and Madych and Nelson [4] showed that MQ-RBF approximation methods converge exponentially as the density of RBFs and their shape parameters increase. Kansa first implemented MQ-RBFs (here referred to as Direct/Differential RBF or DRBF methods) for solving PDEs [5,6]. Since then, DRBF methods have been increasingly used for the solution of elliptic, parabolic and hyperbolic PDEs which govern many engineering problems. In [7–11], practitioners demonstrated that the elliptic PDE solutions using DRBFs converge much faster than those based on polynomial approximations. Mai-Duy and Tran-Cong proposed the idea of using Indirect/Integrated RBFs (IRBFs) for the solution of PDEs [12,13]. Numerical results in [12–19] showed that the integral approach is more accurate than the differential approach. In these works, the authors claimed that because the integration is

* Corresponding author.
E-mail addresses: camminhtri.tien@usq.edu.au (C.M.T. Tien), nam.mai-duy@usq.edu.au (N. Mai-Duy), canh-dung.tran@usq.edu.au (C.-D. Tran), thanh.tran-cong@usq.edu.au (T. Tran-Cong).

http://dx.doi.org/10.1016/j.camwa.2016.09.001
0898-1221/

a smoothing operation and the integrated basis functions are of higher orders, the integral approach has the ability to yield a faster converging solution. In DRBF and IRBF methods, the original unknowns are the RBF coefficients. However, like the Differential Quadrature (DQ) method, these unknowns can be expressed in terms of nodal values of the dependent variable and the calculation is then conducted in the physical space [15].

However, despite the success of RBF methods in many scientific and engineering applications, their accuracy is dependent on a user defined parameter, namely the RBF width or the shape parameter. In this work, it is denoted by β. Numerical experiments indicated that the optimal value of β depends on the function to be interpolated, the configuration of nodal points, the RBF type, and the machine precision [3,4,9,20–23]. The matrix condition of the RBF method grows exponentially with the RBF width. For many problems, e.g. those having smooth solutions, the optimal value of the RBF width is known to be normally large however the corresponding coefficient matrix becomes ill-conditioned. An on-going problem involving the use of RBFs is how to choose the optimal value or even a consistently "good" value of β, which has received a great deal of attention of many researchers. Rippa [21] presented a leave-one-out cross-validation scheme for optimising the shape parameter. For smooth functions, it was shown that without round-off error the highest accuracy for a given number of nodal points is regularly achieved when the RBFs become increasingly flat [8]. Theoretical and computational aspects of increasingly flat RBF interpolations were discussed in [24]. Fornberg and Wright [11] proposed the Contour-Padé algorithm which can stably compute the whole region of the shape parameter on the complex plane. Many different approaches to enhance the stability of DRBF methods have been proposed, for example [23,25–32] and their references therein. For IRBF approaches, Sarra [16] studied the case of global flat IRBFs. It was observed that the even-order IRBFs are generally most accurate and most poorly conditioned for large values of the shape parameter β. Additionally, numerical results in [15,16] showed that the use of higher-order IRBFs can lead to better accuracy. Further discussions about RBF can be found in [33–35] and references therein.

Motivated by the aforementioned works, this paper proposes (i) an easy-to-implement but effective preconditioning technique for Compact IRBF (CIRBF) schemes to alleviate ill-condition problems arising from using large values of β; and (ii) a Combined Compact IRBF (CCIRBF) approximation scheme using high-order IRBFs to enhance the solution accuracy, especially in the large value range of β. Unlike compact schemes previously proposed in [19,36–38], a preconditioning technique is employed here. The present preconditioned CCIRBF scheme is able to stably compute second-order PDE problems with much larger values of β. We derive expressions for evaluation of first- and second-order derivative operators for solving PDE problems and demonstrate the stability and accuracy of the new scheme through various numerical experiments. It should be emphasised that a mesh-free property of RBFs allows lengths between nodes in the stencil to be different. It will be shown that a high level of accuracy is still achieved when CCIRBF stencils are applied to problems with curved boundaries. The strength of RBF methods lies in their ability to deal with scattered data. In the present work, this strength is exploited in the context of Cartesian grid discretisations. It is noted that creating a Cartesian grid is generally much more efficient than creating a finite-element mesh, particularly for domains of non-rectangular shapes. Unlike RBF-DQ methods, our proposed approximations are compact, which helps achieve a high level of accuracy (e.g. avoid the loss of information in the approximation near the curved boundary).

The structure of this paper is organised as follows. Section 2 numerically discusses the condition number of IRBFs over a wide range of β. To enhance the accuracy, a new approximation scheme, CCIRBF, is proposed in Section 3. Following this, a simple preconditioning technique is proposed in Section 4 to retain the accuracy of the CCIRBF when working in the large value range of β. Numerical examples in which the CCIRBF results are compared with some other solutions, where appropriate, are presented in Section 5. Finally, some concluding remarks are given in Section 6.

2. Numerical observations on condition numbers of IRBFs

Several IRBF approximation schemes were previously reported in [12,19,37,38] and they are summarised here for convenience. In IRBF approaches, the MQ function is usually chosen as the basis function

$$G_i(x) = \sqrt{(x - c_i)^2 + a_i^2}, \tag{1}$$

where c_i and a_i are the centre and the width of the ith MQ, respectively. On a stencil, the set of nodal points is taken to be the same as the set of MQ centres. The MQ width is defined as $a_i = \beta h_i$, where β is a positive scalar (the shape parameter) and h_i is the distance between the ith node and its closest neighbour.

For second-order PDEs, the integral approach normally starts with the decomposition of the second-order derivatives of a variable, u, into RBFs

$$\frac{d^2u(\eta)}{d\eta^2} = \sum_{i=1}^{m} w_i G_i(\eta), \tag{2}$$

where $\{G_i(\eta)\}_{i=1}^m$ is the set of RBFs; and $\{w_i\}_{i=1}^m$ is the set of weights/coefficients to be found. Approximate representations for the first-order derivatives and the functions itself are then obtained through the integration processes

Please cite this article in press as: C.M.T. Tien, et al., A numerical study of compact approximations based on flat integrated radial basis functions for second-order differential equations, Computers and Mathematics with Applications (2016), http://dx.doi.org/10.1016/j.camwa.2016.09.001