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Preconditioning of complex linear systems from the Helmholtz equation☆

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ABSTRACT

In this paper, for solving a class of complex linear systems from the Helmholtz equation efficiently, a new splitting preconditioner is established and a real-valued preconditioned iterative method is presented. Spectral properties of the preconditioned matrix are discussed and bound on the eigenvalues of the preconditioned matrix is given. Theorem which provides the dimension of the Krylov subspace methods for the preconditioned iteration method is obtained. The implementation of the preconditioned method is given and the optimal choice of the accelerated parameter is derived. In particular, a more practical way to choose the accelerated parameter is also proposed. Numerical experiments arising from the Helmholtz equation are used to illustrate the performance of the preconditioner, which show the effectiveness and robustness of the new preconditioned GMRES method and demonstrate meshsize-independent and wave number-insensitive convergence behavior.

1. Introduction

In this paper, we consider the iterative solution of the following large, sparse, nonsingular complex system of linear equations

$$\mathscr{M}u = (T + iW)u = c, \quad \mathscr{M} \in \mathbb{C}^{n\times n},\ u, c \in \mathbb{C}^n, \tag{1.1}$$

where $W, T \in \mathbb{R}^{n\times n}$ and W is symmetric positive definite (SPD) and T is symmetric indefinite matrices, $i = \sqrt{-1}$ denotes the imaginary unit. Such linear systems (1.1) arise in numerous applications, such as distributed control problems [1], structural dynamics [2], FFT-based solution of certain time-dependent PDEs [3], molecular scattering, lattice quantum chromo dynamics [4] and so on. For more details about the practical backgrounds of this class of problems, we refer to [5–7] and the references therein.

Let $u = x + iy$ and $c = f + ig$, where $x, y, f, g \in \mathbb{R}^n$. Then as shown in [8,9], the complex linear systems (1.1) can be rewritten as the following two-by-two block real equivalent formulation

$$\mathscr{A}\tilde{z} \equiv \begin{bmatrix} W & T \\ -T & W \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} g \\ -f \end{bmatrix} \equiv b. \tag{1.2}$$

It is noticed in [10,11] that the linear systems (1.2) avoid using complex arithmetic, but the coefficient matrix in (1.2) is doubled in size. The linear system of Equation (1.2) can also be regarded as a special class of generalized saddle-point

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problems [5,12]. Many efficient iterative methods have been proposed to solve the linear systems (1.1) or (1.2), see [7,13]. The Krylov subspace methods for (1.1) or (1.2) are most often used in combinations with appropriate preconditioner, such as preconditioned GMRES [14], PCG methods [15] and QMR [16]. High quality preconditioners are crucial for guaranteeing their accuracy, efficiency, and robustness in actual computations and a general criterion for an efficient preconditioner is that it can be easily implemented and the total CPU time and computational overhead should be reduced, see [17,5,9,8,18,6,19] and references therein.

When both W and T are symmetric positive semi-definite with at least one of them being positive definite, some efficient iterative methods have been proposed. In [20], Bai et al. proposed the modified HSS (MHSS)iteration method, which is much more efficient than the HSS iteration method [21] for the complex symmetric linear systems (1.1) and it is convergent for any positive constant α. In [22], to accelerate the convergence rate of MHSS iteration method, Bai et al. proposed a preconditioned variant of the MHSS (PMHSS). In [23], Hezari et al. presented the preconditioned generalized SOR (PGSOR) iterative method to solve the block two-by-two real equivalent form. Numerical results have shown that the PGSOR iteration method can lead to better computing efficiency than MHSS iteration method.

Whereas if T is an indefinite matrix, the matrix $(\alpha I + T)$ or $(\alpha V + T)$ may be indefinite or singular, and the convergence speeds of the MHSS, PMHSS and PGSOR methods may be converge slowly because the spectral radius of the iteration matrix of the corresponding iteration method may be very close to 1. Thus, the MHSS or PMHSS method may be invalid. Recently, motivated by the ideas suggested in [24,25], Wu [26] developed the HNS iteration method and simplified HNS (SHNS) for solving the complex symmetric linear systems (1.1). To accelerate the convergence rate, Zhang et al. in [27] established a preconditioned SHNS (PSHNS) iteration method and constructed a new preconditioner. The PSHNS iteration can been described as follows.

Algorithm 1.1 (*PSHNS Iteration Method*). Given initial vectors $u^{(0)} \in \mathbb{C}^n$ and positive constant α, for $k = 0, 1, 2 \ldots$ until $u^{(k)}$ converges, compute

$$\begin{cases} (\alpha V + iT)u^{\left(k+\frac{1}{2}\right)} = (\alpha W - TV^{-1}T)u^{(k)} + i\alpha c, \\ (\alpha W + TV^{-1}T)u^{(k+1)} = (\alpha V - iT)u^{\left(k+\frac{1}{2}\right)} - i\alpha c, \end{cases} \tag{1.3}$$

where α is a given positive constant and $V \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix.

The corresponding PSHNS preconditioner is $P_{PSHNS} = \frac{1}{2\alpha}(\alpha V + iT)T^{-1}(\alpha W + TV^{-1}T)$. The PSHNS method (1.3) involves complex arithmetic, it may slow to solve the linear systems (1.1). In this paper, we further consider the solution of linear systems (1.2) with T being indefinite or singular and W being SPD. An efficient splitting preconditioner (ESP) is established and a real-valued preconditioned iterative method is considered. The convergence of the iteration method and some spectral properties of the preconditioned matrix are discussed. This method contains a parameter α, to be chosen by the user. In this paper, the optimal iteration parameter is discussed both in theory and in practice implementation which can result in fast convergence. Eigenvalue distributions of the preconditioned matrix are described and the bounds of the degree of minimal polynomials are obtained. Numerical experiments show that the new preconditioner is competitive and feasible for Krylov subspace methods such as GMRES.

The outline of this paper is as follows. In Section 2, the new preconditioner and the corresponding iteration method for the linear systems (1.2) are presented. The implementation of the new preconditioned GMRES method is given. In Section 3, convergence analyses of the corresponding iteration method are presented. Some spectral properties of the preconditioned matrix and its minimum polynomial are established. In Section 4, the optimal selection of the accelerated parameter is discussed. In addition, a feasible choice of the quasi-optimal parameter in practice is given. In Section 5, numerical experiments are presented to compare the new preconditioner with PSHNS preconditioner. Finally, some brief concluding remarks are given in Section 6.

2. The new preconditioner and its implementation

In this section, we establish an effective splitting preconditioner (ESP) for solving the linear systems (1.2). The ESP is of the form

$$P_\alpha = \begin{bmatrix} \frac{1}{\alpha}W & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \alpha I & T \\ -T & \alpha I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \frac{1}{\alpha}W \end{bmatrix}, \quad \text{with } \alpha > 0. \tag{2.1}$$

In fact, P_α can result in the following matrix splitting

$$\mathscr{A} = P_\alpha - R_\alpha \tag{2.2}$$

with

$$R_\alpha = P_\alpha - \mathscr{A} = \begin{bmatrix} 0 & \frac{1}{\alpha^2}WTW - T \\ 0 & 0 \end{bmatrix}. \tag{2.3}$$