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The blow-up of solutions for m-Laplacian equations with variable sources under positive initial energy ☆

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ABSTRACT

This paper deals with homogeneous Dirichlet boundary value problem to a class of m-Laplace equations with variable reaction

$$\frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{m-2}\nabla u) = u^{q(x)}, x \in \Omega, t > 0,$$

the bounded domain $\Omega \subset R^N (N \geq 1)$ with a smooth boundary. We prove that the weak solutions of the above problems blow up in finite time for all $q^- > m-1 (m \geq 2)$, when the initial energy is positive and initial data is suitably large. This result improves the recent result by Zhou and Yang (2015), which asserts the blow-up of solutions for $N > m$, provided that $q^+ < \frac{Nm-m-N}{N-m}$.

1. Introduction

We consider the following m-Laplace equations with a variable source

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{m-2}\nabla u) = u^{q(x)}, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t \geq 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^N (N \geq 1)$ is a bounded domain, $\partial\Omega$ is Lipschitz continuous and $u_0 \geq 0$, $m \geq 2$.

Model (1.1) proposed by M. *Ružička* may describe some properties of electro-rheological fluids which change their mechanical properties dramatically when an external electric field is applied [1,2]. Another important application is the image processing where the anisotropy and nonlinearity of the diffusion operator are used to underline the borders of the distorted image and to eliminate the noise [3,4]. For more physical background, we refer the readers to papers [5,6].

Our main aim in this paper is to study the blow-up phenomenon, which means that there exists a solution $u(x, t)$ that becomes unbounded in finite time. In recent years, more and more people pay attention to the blow-up of solutions, we can see the survey paper [7]. In [8], when $q(x) = q \geq m-1$, the solutions of Problem (1.1) blow up in finite time if the initial $u_0(x)$ is sufficiently large or $E(0) < 0$, where $E(t)$ is an energy function. It is well known that parabolic equations with

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variable nonlinearity may possess, for certain ranges of the exponents, the localization properties which are intrinsic for the solutions of nonlinear equation with constant nonlinearity such as vanishing in a finite time, finite speed of propagation on disturbances from the data or waiting time phenomena, see [9,10]. Up to our knowledge, there are less papers to study the blow-up of solutions for parabolic equations with variable exponent source, see [11–15].

When $m = 2, q^- > 2$, Khelghati [14] et al. have proved that the solutions of Problem (1.1) blow up in finite time with arbitrary positive initial energy and suitable large initial values. The authors in [15] have proved that the solutions of Problems (1.1) blow up by constructing a control function and applying suitable embedding theorems, which is on the following conditions

$$E(0) < E_1, \qquad \|\nabla u_0\|_m > \alpha_1,$$

where E_1 and α_1 are positive constants. In addition, the function $q(x)$ must satisfy the following conditions

$$\begin{cases} 1 < q^+ < +\infty, & N \le m, \\ 1 < q^+ \le \dfrac{Nm + m - N}{N - m}, & N > m. \end{cases}$$

Motivated by the above work, we establish blow-up of solutions for Problems (1.1) with arbitrary positive initial energy and suitable large initial values, provided that $q^- > m - 1$ $(m \ge 2)$ and $N \ge 1$, which extends the result in [14,15]. In addition, we also prove that the nonnegative solutions of Problem (1.1) must blow up in finite time under negative initial energy.

Let $q(x)$ satisfy the following conditions:

$$1 < q^- = \inf_{x\in\Omega} q(x) \leqslant q(x) \leqslant q^+ = \sup_{x\in\Omega} q(x) < \infty, \tag{1.2}$$

$$\forall z \in \Omega,\ \xi \in \Omega,\ |z - \xi| < 1, \quad |q(z) - q(\xi)| \leqslant \omega(|z - \xi|), \tag{1.3}$$

where

$$\lim_{\tau\to 0^+} \omega(\tau) \ln \frac{1}{\tau} = C < +\infty.$$

2. Blow up in finite time

In this section, we will prove that the blow-up of solutions to Problem (1.1) with arbitrary positive energy and suitable initial data. As it is well known that degenerate equations do not have classical solutions, we give a precise definition of the weak solution.

Definition 2.1. A function $u(x, t) \in L^\infty(\Omega \times (0, T)) \cap L^m(0, T; W_0^{1,m}(\Omega))$, $u_t \in L^2(0, T; L^2(\Omega))$ is called weak solution of Problem (1.1), if and only if the equality

$$\int_\Omega u\varphi|_{t_1}^{t_2} \mathrm{d}x - \int_{t_1}^{t_2} \int_\Omega u\varphi_t \mathrm{d}x\mathrm{d}\tau + \int_{t_1}^{t_2} \int_\Omega |\nabla u|^{m-2}\nabla u \cdot \nabla\varphi_t = \int_{t_1}^{t_2} \int_\Omega u^{q(x)}\varphi \mathrm{d}x\mathrm{d}\tau$$

holds for all $0 < t_1 < t_2 < T$, where $\varphi \in C^{1,1}(\bar{\Omega} \times [0, T])$ such that $\varphi(x, T) = 0$ and $\varphi(x, t) = 0,\ x \in \partial\Omega \times [0, T]$.

The existence and uniqueness of Problem (1.1) were discussed in paper [12]. The main method that proves the blow-up of solutions is based on the calculation of the energy function and concavity argument developed by Levine [16].

Lemma 2.1 ([16]). *Suppose that a positive, twice-differentiable function $\theta(t)$ satisfies the inequality*

$$\theta''(t)\theta(t) - (1 + \beta)\theta'(t)^2 \ge 0, \quad t > 0,$$

where $\beta > 0$ is some constant. If $\theta(0) > 0$ and $\theta'(0) > 0$, then there exists $0 < T_1 < \frac{\theta(0)}{\beta\theta'(0)}$ such that $\theta(t)$ tends to infinity as $t \to T_1$.

Here, we have our main result

Theorem 2.1. *Let $u(x, t)$ be a solution of Problem (1.1) in a bounded domain $\Omega \subset R^N (N \ge 1)$. Then for all $q^- > m - 1$ and $E(u_0) > 0$, the solution $u(x, t)$ blows up in finite time provided that*

$$\|u_0\|_2^2 \ge \max\left\{ C_1 E^{\frac{2}{q^- + 1}}(u_0), C_2 E^{\frac{2}{q^+ + 1}}(u_0) \right\},$$

where

$$C_1 = \left(\frac{m(q^- + 1)}{q^- + 1 - m}\right)^{\frac{2}{q^- + 1}} \left[\frac{(q^+ - 1)(q^- + 1)|\Omega|}{(q^- - 1)(q^+ + 1)}\right]^{\frac{q^- - 1}{q^- + 1}}, \qquad C_2 = \left(\frac{m(q^+ + 1)|\Omega|^{\frac{q^+ - 1}{2}}}{q^- + 1 - m}\right)^{\frac{2}{q^+ + 1}}. \tag{2.1}$$