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On the finite element method for a nonlocal degenerate parabolic problem

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ABSTRACT

The aim of this paper is the numerical study of a class of nonlinear nonlocal degenerate parabolic equations. The convergence and error bounds of the solutions are proved for a linearized Crank–Nicolson–Galerkin finite element method with polynomial approximations of degree $k \geq 1$. Some explicit solutions are obtained and used to test the implementation of the method in Matlab environment.

1. Introduction

In this work, we study parabolic problems with nonlocal nonlinearity of the following type

$$\begin{cases} u_t - \left(\int_\Omega u^2(x,t)dx \right)^\gamma \Delta u = f(x,t), & (x,t) \in \Omega \times]0,T], \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times]0,T], \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{cases} \tag{1}$$

where Ω is a bounded open domain in $\mathbb{R}^d$, $d = 1, 2, 3$, γ is a real constant, and f and u_0 are continuous integrable functions.

In 1996, Chipot and Lovat [1] proposed the equation

$$u_t - a\left(\int_\Omega u\, dx \right) \Delta u = f \tag{2}$$

to model the density of a population subject to spreading or heat propagation, and proved the existence and uniqueness of weak solutions. Since then, the existence, uniqueness and asymptotic behaviour of weak and strong solutions of parabolic equations and systems with nonlocal diffusion terms, have been widely studied (see, for example, [2–4] and their references).

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Concerning the numerical treatment of these equations we refer to some relevant works. Ackleh and Ke [5] studied the problem

$$\begin{cases} u_t = \dfrac{1}{a(\int_\Omega u\,dx)}\Delta u + f(u), \quad (x,t) \in \Omega \times]0,T], \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times]0,T], \\ u(x,0) = u_0(x), \quad x \in \overline{\Omega}, \end{cases}$$

with $a(\xi) > 0$ for all $\xi \neq 0$, $a(0) \geq 0$ and f Lipschitz-continuous satisfying $f(0) = 0$. In addition to the proof of the existence and uniqueness of solutions to this problem and the establishment of conditions on u_0 for the extinction in finite time and for the persistence of solutions, they also made some numerical simulations with a finite difference scheme in one dimension and a finite volume discretization in two space dimensions. In 2009, Bendahmane and Sepúlveda [6] used the model

$$\begin{cases} (u_1)_t - a_1\left(\displaystyle\int_\Omega u_1\,dx\right)\Delta u_1 = -\sigma(u_1,u_2,u_3) - \mu u_1, \\ (u_2)_t - a_2\left(\displaystyle\int_\Omega u_2\,dx\right)\Delta u_2 = \sigma(u_1,u_2,u_3) - \gamma u_2 - \mu u_2, \\ (u_3)_t - a_3\left(\displaystyle\int_\Omega u_3\,dx\right)\Delta u_3 = \gamma u_2, \end{cases} \tag{3}$$

to investigate the propagation of an epidemic disease, in a physical domain $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$). They established the existence of discrete solutions to a finite volume scheme and its convergence to the weak solution of the PDE. In [7], the authors proved the optimal order of convergence for a linearized Euler–Galerkin finite element method for a nonlocal system with absorption,

$$\begin{cases} u_t - a_1(l_1(u), l_2(v))\Delta u + \lambda_1|u|^{p-2}u = f_1(x,t), \quad (x,t) \in \Omega \times]0,T], \\ v_t - a_2(l_1(u), l_2(v))\Delta v + \lambda_2|v|^{p-2}v = f_2(x,t), \quad (x,t) \in \Omega \times]0,T], \end{cases} \tag{4}$$

and presented some numerical results. In [8], Robalo et al. obtained approximate numerical solutions for a nonlocal reaction–diffusion system, in a domain with moving boundaries, of the type

$$\begin{cases} u_t - a_1\left(\displaystyle\int_{\Omega(t)} v\,dx\right)u_{xx} = f_1(x,t), \quad (x,t) \in \hat{Q}, \\ v_t - a_2\left(\displaystyle\int_{\Omega(t)} u\,dx\right)v_{xx} = f_2(x,t), \quad (x,t) \in \hat{Q}, \end{cases} \tag{5}$$

where $\hat{Q} = \{(x,t) \in \mathbb{R}^2 : \alpha(t) < x < \beta(t), 0 < t < T\}$, with a Matlab code based on the moving finite element method (MFEM) with high degree local approximations. Almeida et al. [9,10], established the convergence and error bounds of the fully discrete solutions for a class of nonlinear equations and for systems of reaction–diffusion nonlocal type with moving boundaries, using a linearized Crank–Nicolson–Galerkin finite element method with polynomial approximations of any degree.

In this paper, we analyse a different diffusion term, dependent on the L_2-norm of the solution. In most of the previous papers, it is assumed that the diffusion term is bounded, with $0 < m \leq a(s) \leq M < \infty$, $s \in \mathbb{R}$, and so the problem is always nondegenerate. Here, we study a case where the diffusion term could be zero or infinity. Problem (1) was studied in [11], where the authors proved the existence of weak solutions for $t \in [0,T]$ and the existence of a positive instant t^* such that these solutions are unique and classical for $t \in [0,t^*]$. In [11], the asymptotic behaviour of the solutions as time increases, was also studied.

This work is concerned with the study of the convergence of a total discrete solution using a Crank–Nicolson–Galerkin finite element method and the use of this method to analyse the behaviour of the weak solutions. To the best of our knowledge, these results are new for nonlocal reaction–diffusion equations with this type of diffusion term. The remaining of this paper is organized as follows. In Section 2, we formulate the problem and recall some useful definitions and lemmas. In Section 3, we define and prove the convergence of the semidiscrete solution. Section 4 is devoted to the definition and proof of the convergence of a fully discrete solution. In Section 5, we obtain some explicit solutions and analyse their behaviour, and then we use the deduced explicit solutions to simulate some examples in Section 6. Finally, in Section 7, we draw some conclusions.

2. Statement of the problem

Let Ω be a bounded open domain in $\mathbb{R}^d$, $d = 1, 2, 3$, with Lipschitz-continuous boundary $\partial\Omega$, and T an arbitrary positive finite instant. We consider the problem of finding the function $u(x,t)$ which satisfies the following conditions

$$\begin{cases} u_t - a(u)\Delta u = f(x,t), \quad (x,t) \in \Omega \times]0,T], \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times]0,T], \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{cases} \tag{6}$$

where $a(u) = \left(\int_\Omega u^2(x,t)dx\right)^\gamma$ with $\gamma \in \mathbb{R}$ and f and u_0 are continuous integrable functions.

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