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Existence and multiplicity of solutions for a generalized Choquard equation[☆]

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ABSTRACT

In this paper, we study a generalized Choquard equation

$$-\Delta u+V(x)u=\Big(\int_{\mathbb{R}^N}\frac{Q(y)F(u(y))}{|x-y|^{\mu}}dy\Big)Q(x)f(u),\quad u\in H^1(\mathbb{R}^N),$$

where $0<\mu<N$, V and Q are linear and nonlinear potentials, and F is the primitive function of f. When the potentials are periodic and f is odd or even, we find infinitely many geometrically distinct solutions using the method of Nehari manifold and index theory. When the potentials are generalized asymptotically periodic, we show the existence of ground states by means of the method of Nehari manifold and concentration compactness principle.

1. Introduction and statement of the main result

The Choquard equation

$$-\varDelta u+u=\Big(\int_{\mathbb{R}^3}\frac{|u(y)|^2}{|x-y|}dy\Big)u,\quad u\in H^1(\mathbb{R}^3),\tag{1.1}$$

which was proposed by Choquard in 1976, and can be described as an approximation to Hartree–Fock theory of a one-component plasma, see [1]. It was also proposed by Penrose in [2] as a model for the self-gravitational collapse of a quantum mechanical wave function. In this context, problem (1.1) is usually called the nonlinear Schrödinger–Newton equation. In [1], Lieb proved the existence and uniqueness of a minimizer to problem (1.1) by using symmetric decreasing rearrangement inequalities. Later, in [3], Lions showed the existence of infinitely many radially symmetric solutions of (1.1). Further results for related problems we refer to [4–8] and references therein.

In [9], Ma and Zhao considered the generalized Choquard equation

$$-\varDelta u+u=\Big(\int_{\mathbb{R}^N}\frac{|u(y)|^p}{|x-y|^{\mu}}dy\Big)|u|^{p-2}u,\quad u\in H^1(\mathbb{R}^N),\tag{1.2}$$

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for $p \geq 2$ and $N \geq 3$. Under the condition that a certain set of real numbers N, μ, and p is nonempty, they proved that every positive solution of (1.2) is radially symmetric and monotone decreasing about some point. Under the same assumption, Cingolani, Clapp and Secchi [10] obtained some existence and multiplicity results in the electromagnetic case, and established the regularity and some decay at infinity of ground states for (1.2). Moroz and Van Schaftingen [11] eliminated this restriction and they showed the regularity, positivity and radial symmetry of ground states in the optimal range of parameters. Later, Clapp and Salazar [12] considered Eq. (1.2) with the linear potential satisfying a certain symmetry assumptions on unbounded domain Ω and some decay conditions at infinity, and they obtained the existence of a positive solution and multiple sign changing solutions. Moroz and Van Schaftingen [13] treated (1.2) with general nonlinearity in the spirit of Berestycki and Lions and proved the existence of ground states. On the other hand, some people have studied the semi-classic states, and there are many results about the existence and concentration of solutions for (1.2). See [4,5,14–17]. Recently, Alves and Yang [17] considered Eq. (1.2) with general potentials and nonlinearity that

$$-\epsilon^2\Delta u+V(x)u=\epsilon^{\mu-N}\Big(\int_{\mathbb{R}^N}\frac{Q(y)F(u(y))}{|x-y|^\mu}dy\Big)Q(x)f(u),\quad u\in H^1(\mathbb{R}^N). \tag{1.3}$$

Under suitable assumptions of V, Q, and f, they established a new concentration behavior of solutions by variational methods.

For the Schrödinger equation

$$-\Delta u+V(x)u=f(x,u),\quad u\in H^1(\mathbb{R}^N), \tag{1.4}$$

there are many results under various conditions of V and f. When V and f are periodic in x, and f satisfies a certain monotone condition and superquadratic condition in u, based on the method of Nehari manifold Szulkin and Weth [18,19] showed the existence of ground states and the multiplicity of geometrically distinct solutions.

By motivation of these works [17–19], we are interested in the existence and multiplicity of solutions for the generalized Choquard equation

$$-\Delta u+V(x)u=\Big(\int_{\mathbb{R}^N}\frac{Q(y)F(u(y))}{|x-y|^\mu}dy\Big)Q(x)f(u),\quad u\in H^1(\mathbb{R}^N), \tag{1.5}$$

where V, Q and f are continuous real functions and F is the primitive function of f. To the best of our knowledge, there is no result about geometrically distinct solutions, and we shall find infinitely many geometrically distinct solutions for (1.5) with periodic potentials. In addition, we also consider the existence of ground states for (1.5) in which the potentials are generalized asymptotically periodic.

For the potentials, assume that:

(VQ_1) $V, Q\in C(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, $\inf_{\mathbb{R}^N} V>0$ and $\inf_{\mathbb{R}^N} Q>0$.

For the nonlinearity f, suppose that $f\in C(\mathbb{R})$ and satisfies the following conditions:

(H_1) $|f(s)|\leq a(|s|^{q_1-1}+|s|^{q_2-1})$, for some $a>0$ and $2-\frac{\mu}{N}<q_1\leq q_2<\frac{2^*}{2}(2-\frac{\mu}{N})$,

(H_2) $s\mapsto f(s)$ is increasing on $(-\infty,0)$ and $(0,+\infty)$,

(H_3) $\frac{F(s)}{|s|}\to+\infty$ as $|s|\to+\infty$, where $F(s)=\int_0^s f(t)dt$.

Firstly, we consider the periodic case:

(VQ_2) V and Q are 1-periodic in each component x_j with $x=(x_1,x_2,\dots,x_N)$,

Let $\star$ be the action of $\mathbb{Z}^N$ on $H^1(\mathbb{R}^N)$ given by

$$(k\star u)(x):=u(x-k),\quad k\in\mathbb{Z}^N.$$

From (VQ_2) it follows that if u_0 is a solution of (1.5), then so is $k\star u_0$ for all $k\in\mathbb{Z}^N$. Set

$$\mathcal{O}(u_0):=\{k\star u_0:k\in\mathbb{Z}^N\}.$$

$\mathcal{O}(u_0)$ is called the orbit of u_0 with respect to the action of $\mathbb{Z}^N$. Two solutions u_1, u_2 of (1.5) are said to be geometrically distinct if $\mathcal{O}(u_1)\neq\mathcal{O}(u_2)$.

Theorem 1.1. *Let* (VQ_1), (VQ_2) *and* (H_1)–(H_3) *hold. If f is odd or even in u, then the problem* (1.5) *admits infinitely many pairs $\pm u$ of geometrically distinct solutions.*

Below we consider the asymptotically periodic case. Let $\mathcal{F}$ be the class of functions $h\in L^\infty(\mathbb{R}^N)\cap C(\mathbb{R}^N)$ such that, for every $\epsilon>0$ the set $\{x\in\mathbb{R}^N:|h(x)|\geq\epsilon\}$ has finite Lebesgue measure.

(VQ_3) there exist a constant $b>0$ and functions $V_p, Q_p\in L^\infty(\mathbb{R}^N)$, 1-periodic in x_i, $1\leq i\leq N$, such that

(i) $V-V_p\in\mathcal{F}$, $Q-Q_p\in\mathcal{F}$,

(ii) $V(x)\leq V_p(x)$, and $b\leq Q_p(x)\leq Q(x)$, for all $x\in\mathbb{R}^N$.

Theorem 1.2. *Let* (VQ_1), (VQ_3) *and* (H_1)–(H_3) *hold. Then the problem* (1.5) *possesses a ground state.*

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