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Chemotaxis-driven pattern formation for a reaction–diffusion–chemotaxis model with volume-filling effect

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ABSTRACT

In this paper we analytically and numerically investigate the emerging process of pattern formation for a reaction–diffusion–chemotaxis model with volume-filling effect. We first apply globally asymptotic stability analysis to show that the chemotactic flux is the key mechanism for pattern formation. Then, by weakly nonlinear analysis with multiple scales and the adjoint system theory, we derive the cubic and the quintic Stuart–Landau equations to describe the evolution of the amplitude of the most unstable mode, and thus the analytical approximate solutions of the patterns are obtained. Next, we present the selection law of principal wave mode of the emerging pattern by considering the competition of the growing modes, and for this we deduce the change rule of the most unstable mode and the coupled ordinary differential equations that indicates the significant nonlinear interaction of two competing modes. Finally, in the subcritical case we clarify that there exists the phenomenon of hysteresis, which implies the existence of large amplitude pattern for the bifurcation parameter values smaller than the first bifurcation point. Therefore, we answer the open problems proposed in the known references and improve some of results obtained there. All the theoretical results are tested against the numerical results showing excellent qualitative and good quantitative agreement.

1. Introduction

Chemotaxis is a motion of organisms induced by variations in the concentration of chemicals. It is observed that aggregates form when organisms move up or down *the concentration gradient*. Like the Turing mechanism, it is considered an important mechanism for many biological pattern formations such as the propagation of traveling band of bacteria toward the oxygen [1,2], the outward propagation of concentric ring waves by *motile cells of Escherichia coli* [3–5], and the spiral wave patterns during the aggregation of *Dictyostelium discoideum* [6,7]. Patlak and Keller and Segel did the pioneering works to the mathematical modeling of chemotaxis in 1953 [8] and in 1970 [9,10], respectively. Based on the Keller–Segel model, a variety of chemotaxis models have been proposed to describe the chemotactic aggregation process. Among them, one includes a so-called volume-filling chemotaxis so that arbitrarily high cell densities can be precluded by setting an impassable

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threshold value for cell density in [11]. In [12,13], Wang and Hillen further developed this idea for generic cell types, and then a generalized form of volume-filling chemotaxis model of [11,12] reads

$$\begin{cases} u_t = \nabla\cdot(D(1-u)^{-\alpha}\nabla u - \chi u(1-u)^{\beta}\nabla v) + \mu u(1-u/u_c), & x\in\Omega,\ t>0,\\ v_t = \Delta v - v + u, & x\in\Omega,\ t>0,\\ \nabla u\cdot\nu = \nabla v\cdot\nu = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x), \qquad v(x,0)=v_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $D(1-u)^{-\alpha}$ is the cell density-dependent diffusion coefficient and the function $u(1-u)^{\beta}$ represents the chemotactic sensitivity, here the number 1 is defined as *crowding capacity* i.e., the maximal cell numbers that can be accommodated in a unit volume of space; $(x,t)\in\Omega\times[0,+\infty)$ and Ω is a bounded convex domain in $\mathbb{R}^N$ ($N=1,2$ and 3) with smooth boundary $\partial\Omega$; ν denoting the outward unit normal vector on $\partial\Omega$; $D>0$ and $\alpha,\beta\in\mathbb{R}$ are constants; $\mu>0$ is the intrinsic growth rate of the cell and u_c stands for the carrying capacity with $0<u_c<1$; $\chi>0$ is called the chemotactic coefficient.

The more detailed information of (1.1) can be found in [14,15], such as its derivation and the study of various models related to (1.1). In [14,15], authors investigated the global existence of classical solutions, local and global bifurcation of steady state and its stability, the boundedness and existence of non-constant steady states of (1.1) with initial data (u_0,v_0) satisfying

$$(u_0,v_0)\in[W^{1,\infty}(\Omega)]^2 \quad\text{and}\quad 0\le u_0(x)<1, \qquad v_0(x)\ge 0, \quad x\in\overline{\Omega}. \tag{1.2}$$

In this paper, based on the results established in [14,15], we shall prove that chemotaxis is the key factor of pattern formation and further investigate the emerging process and the shape of the patterns. This interesting result of the effect from chemotaxis on the dynamics was also found in many references for various models with chemotaxis, for example, see [16,17,12,18]. By using the weakly nonlinear bifurcation analysis with multiple scales and Fredholm theory, we derive the Stuart–Landau equation to describe the evolution of the amplitude of spatiotemporal pattern. Thus explicit formulas for the spatiotemporal pattern (*non-constant classical solutions*) and the stationary pattern (*stable non-constant steady state*) of (1.1)–(1.2) are obtained, which perfectly coincide with the corresponding results obtained in [14]. We also discuss the competition of unstable modes away from the threshold value k_c when the bifurcation parameter χ is large enough. To illustrate the nonlinear interaction of the amplitudes of two competing modes, we deduce a coupled ordinary differential equation whose phase diagram displays how the initial amplitude affects the shape of stationary pattern. The selection law of the wave modes of the stationary patterns is found. For the subcritical case both the theoretical analysis and the bifurcation diagram verify the coexistence of stable steady states in the system (1.1) and the existence of large amplitude pattern before the positive uniform steady state loses its stability. This answers the open question in [19,20].

This paper is organized as follows. In Section 2, we shall show that the pattern formation is caused by chemotaxis and give the critical wave number and the critical value of the bifurcation parameter. In Section 3, following the approach presented by Gambino et al. in [21,22], we derive the Stuart–Landau equations to capture the evolution of the amplitude of the first admissible mode both in the case of supercritical and subcritical bifurcation. Accordingly, the analytical approximation of spatiotemporal patterns are obtained. Section 4 is devoted to discuss the stationary pattern of the supercritical case. Section 5 is to consider the stationary pattern of the subcritical case. Conclusion and problems for further study are presented in Section 6. Finally, Appendices A and B give the details of the derivation of the quintic Stuart–Landau equation and the equations of the amplitudes of the two unstable modes.

Throughout this paper, we will always assume that

$$\alpha+\beta>1 \tag{1.3}$$

and in one dimensional case, we take $\Omega=[0,l]$ with $l>0$.

2. Chemotaxis-driven instability

To verify that the pattern formation of the system (1.1)–(1.2) is driven by chemotaxis, we first give an important lemma.

Lemma 2.1 ([14, Theorem 2.1]). *Let (u_0,v_0) fulfill (1.2), and α and β satisfy (1.3). Then the problem (1.1)–(1.2) has a global classical solution (u,v). Moreover, there exists a constant $\delta>0$ such that*

$$0\le u(x,t)\le 1-\delta, \qquad 0\le v(x,t)\le 1-\delta, \quad \textit{for all } (x,t)\in\Omega\times(0,\infty). \tag{2.1}$$

Then we present a result of the kinetic system of (1.1)–(1.2)

$$\begin{cases} \dfrac{du}{dt} = \mu u\left(1-\dfrac{u}{u_c}\right),\\ \dfrac{dv}{dt} = u - v,\\ u(0)=u^0, \qquad v(0)=v^0, \end{cases} \tag{2.2}$$