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# On the generation of arbitrage-free stock price models using Lie symmetry analysis

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ABSTRACT

In Bell and Stelljes (2009) a scheme for constructing explicitly solvable arbitrage-free models for stock prices is proposed. Under this scheme solutions of a second-order $(1+1)$-partial differential equation, containing a rational parameter $p$ drawn from the interval $[1/2, 1]$, are used to generate arbitrage-free models of the stock price. In this paper Lie symmetry analysis is employed to propose candidate models for arbitrage-free stock prices. For all values of $p$, many solutions of the determining partial differential equation are constructed algorithmically using routines of Lie symmetry analysis. As such the present study significantly extends the work by Bell and Stelljes who found only two arbitrage-free models based on two simple solutions of the determining equation, corresponding to $p = 1/2$ and $p = 1$.



## 1. Introduction

An important constituent in the valuation of options and other derivatives is the stock price. In the classical Black–Scholes model [1] the stock price $S_t$ is assumed to follow an Itô process described by the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{1.1}$$

where $\mu$ and $\sigma$ are two parameters representing the drift and volatility of the stock, respectively, and $W_t$ is a standard Wiener process. The Black–Scholes stock price model (1.1) belongs to a class of solvable arbitrage-free models, i.e. models for which the expected value of $S_t$ at any time $t$ is precisely the future value at time $t$ of a risk-free bond with present value $S_0$. As a result of this feature (1.1) leads to the well-known Black–Scholes formula for determining the value of a European call option [1]. In fact "arbitrage-freeness" is an essential feature in stock price models. Unfortunately, it is not always inherent in alternative models of the stock price.

Bell and Stelljes [2] describe a method for constructing explicitly solvable arbitrage-free models for stock prices. The method is based on the following solvable stochastic Bernoulli equation of *Stratonovich* type

$$d\widetilde{S}_t = \mu \widetilde{S}_t + \sigma \widetilde{S}_t^p \circ dW_t, \tag{1.2}$$

where $p$ denotes a rational number in the interval $[\frac{1}{2}, 1]$. The solution of (1.2) (see [2] and the references therein) is

$$\widetilde{S}_t = e^{rt} \left\{ (1-p)\sigma \int_0^t e^{r(p-1)u} dW_u + \widetilde{S}_0^{1-p} \right\}^{1/(1-p)}. \tag{1.3}$$

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The process $\widetilde{S}_t$ does not generally satisfy the arbitrage-free condition and hence is not a feasible model for stock price. However, $S_t \equiv G(\widetilde{S}_t, t)$ does satisfy the arbitrage-free condition provided that $G$ solves the second-order partial differential equation

$$G_t + \left(rs + \frac{p\,\sigma^2 s^{2p-1}}{2}\right) G_s + \frac{\sigma^2 s^{2p}}{2} G_{ss} = r\,G \tag{1.4}$$

and that there exists $n$ such that for every $T > 0$

$$\sup_{0\le t\le T} |G_s(s,t)| \le C|s|^n \tag{1.5}$$

where $C$ is a constant depending only on $T$.

In [2] two values of $p$, namely $p = \frac{1}{2}$ and $p = 1$, are identified in which cases Eq. (1.4) is tractable. The former case gives rise to a solvable version of the Cox–Ross model [3] and the latter to the Black–Scholes model [1]. In the case $p = 1$,

$$G(s,t) = s\,e^{-\sigma^2 t/2} \tag{1.6}$$

is found to solve (1.4) and to satisfy the regularity condition (1.5). Therefore $S_t = G(\widetilde{S}_t, t)$, with $\widetilde{S}_t$ defined in (1.3), furnishes an arbitrage-free stock price model for $p = 1$. Similarly for $p = \frac{1}{2}$

$$G(s,t) = s + \frac{\sigma^2}{4\,r} \tag{1.7}$$

solves (1.4) and satisfies the regularity condition (1.5). Accordingly, the resulting arbitrage-free stock price model $S_t = G(\widetilde{S}_t, t)$ is obtained from (1.7) and (1.3).

The aim of this paper is to investigate Eq. (1.4) for all values of $p$ for which the equation is tractable. From the point of view of Lie symmetry analysis this coincides with values of $p$ for which the equation admits a nontrivial symmetry Lie algebra. We have determined that for each value of $p$ Eq. (1.4) admits a rich symmetry group akin to the group admitted by the Black–Scholes equation or the heat equation [4]. Furthermore, we have exploited the admitted one-parameter Lie point symmetries and routines of Lie symmetry analysis to construct solutions of (1.4) as invariant solutions and by transformation of known solutions.

The paper is organised as follows. In Section 2, we introduce elements of Lie symmetry analysis of differential equations. Determination of Lie point symmetries admitted by Eq. (1.4) is done in Section 3. In Section 4 we use the admitted symmetries to construct several exact solutions of (1.4) for all rational values of $p$. We present concluding remarks in Section 5.

## 2. Preliminaries of Lie symmetry analysis

Lie symmetry analysis is one of the most powerful methods for finding analytical solutions of differential equations. It has its origins in studies by the Norwegian mathematician Sophus Lie who began to investigate continuous groups of transformations that leave differential equations invariant. Accounts of the subject and its application to differential equations are covered in many books [5–12].

Central to methods of Lie symmetry analysis is invariance of a differential equation under a continuous group of transformations. Consider a one-parameter Lie group of point transformations in infinitesimal form

$$\begin{aligned}\tilde{x} &= x + \varepsilon\,\xi(x,t,u) + O(\varepsilon^2)\\ \tilde{t} &= t + \varepsilon\,\tau(x,t,u) + O(\varepsilon^2)\\ \tilde{u} &= u + \varepsilon\,\eta(x,t,u) + O(\varepsilon^2)\end{aligned} \tag{2.1}$$

depending on a continuous parameter $\varepsilon$. This transformation is characterised by its infinitesimal generator,

$$X = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \eta(x,t,u)\partial_u. \tag{2.2}$$

The corresponding finite transformations are obtained by exponentiating or by solving the *Lie equations*

$$\frac{d\widetilde{x}}{d\varepsilon} = \xi(\widetilde{x},\widetilde{t},\widetilde{u}), \qquad \frac{d\widetilde{t}}{d\varepsilon} = \tau(\widetilde{x},\widetilde{t},\widetilde{u}), \qquad \frac{d\widetilde{u}}{d\varepsilon} = \eta(\widetilde{x},\widetilde{t},\widetilde{u}) \tag{2.3}$$

subject to the initial conditions

$$(\widetilde{x},\widetilde{t},\widetilde{u})|_{\varepsilon=0} = (x,t,u). \tag{2.4}$$

A general $(1+1)$-partial differential equation with one dependent variable $u$ and two independent variables $(x,t)$,

$$\Delta(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt}) = 0 \tag{2.5}$$