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Special least squares solutions of the quaternion matrix equation $AXB + CXD = E$

Fengxia Zhang [a,*], Weisheng Mu [a,b], Ying Li [a], Jianli Zhao [a]

[a] *College of Mathematical Sciences, Liaocheng University, Liaocheng 252059, China*

[b] *College of Mathematics and Science, Shanghai Normal University, Shanghai 200234, China*

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ABSTRACT

In this paper, by using the real representations of quaternion matrices, the particular structure of the real representations of quaternion matrices, the Kronecker product of matrices and the Moore–Penrose generalized inverse, we obtain the expressions of the minimal norm least squares solution, the pure imaginary least squares solution, and the real least squares solution for the quaternion matrix equation $AXB+CXD = E$, respectively. Our resulting formulas only involve real matrices, and therefore are simpler than those reported in Yuan (2014). The corresponding algorithms only perform real arithmetic which also consider the particular structure of the real representations of quaternion matrices, and therefore are very efficient and portable. Numerical examples are provided to illustrate the efficiency of our algorithms.

1. Introduction

In the study of theory and numerical computations of quaternionic quantum theory, in order to well understand the perturbation theory [1], theoretical discussions [2–4], and experimental proposals [5–7] underlying the quaternionic formulations of the Schrödinger equation and so on, we often meet problems of approximate solutions of quaternion problems, such as approximate solutions of the quaternion linear equation $AXB \approx E$ which is appropriate when there is an error in matrix E, i.e., quaternionic least squares (QLS) problem. Due to the extensive applications of the quaternion matrix equations and their least squares solutions in computer science, quantum physics, statistic, signal and color image processing, rigid mechanics, quantum mechanics, control theory, field theory and so on [8–13], many researchers are interested in them. There are some valuable results on the QLS problem. For example, by using the matrix decompositions, the complex representations of quaternion matrices, the Moore–Penrose generalized inverse and the Kronecker product of matrices, Yuan et al. derived the expression of Hermitian solution for the matrix nearness problem associated with the quaternion matrix equation $AXA^H + BYB^H = C$ [14], the least squares Hermitian solution of the quaternion matrix equation $(AXB, CXD) = (E, F)$ with the least norm [15] and the special least squares solutions of the quaternion matrix equation $AX = B$ [16]. By means of the complex representations of quaternion matrices, Jiang et al. studied algebra algorithms for the QLS problem [17], the QLS eigenproblem [18] and the QLS problem with constraints [19] in quaternionic quantum theory, and obtained some theoretical results. Moreover, by the real representations of quaternion matrices, the QLS problem in quaternionic quantum theory in [20] was reconsidered, and the authors derived an operable iterative method called

* Corresponding author.
E-mail address: mathxia@yahoo.com (F. Zhang).

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LSQR-Q algorithm to find the minimum-norm solution of the QLS problem. According to this algorithm, Ling et al. gave an iterative algorithm for finding Hermitian tridiagonal solution with the least norm to the QLS problem [21].

Consider the generalized Sylvester matrix equation $AXB + CXD = E$, where A, B, C, D are given matrices of suitable sizes, and X is an unknown matrix of suitable size over real number field, complex number field or quaternion field. When B and C are identity matrices, it reduces to the well-known Sylvester equation. When C and D are identity matrices, it reduces to the well-known Stein equation. Being well-known in both pure and applied mathematics, it has been investigated extensively [22–29]. In [29], by using the complex representations of quaternion matrices, the Kronecker product of matrices and the Moore–Penrose generalized inverse, the authors obtained the expressions of the minimal norm least squares solution, the pure imaginary least squares solution and the real least squares solution for the quaternion matrix equation $AXB + CXD = E$, and provided several numerical examples to illustrate the efficiency of their methods. Motivated by the above work, we will study these problems again by using the real representations of quaternion matrices, the particular structure of the real representation matrices, the Kronecker product of matrices and the Moore–Penrose generalized inverse. Our resulting formulas only involve real matrices, and therefore are simpler than those reported in [29].

Throughout this paper, let R be the real number field, C the complex number field, and $Q = R \oplus Ri \oplus Rj \oplus Rk$ the quaternion field, where $ij = -ji = k, i^2 = j^2 = k^2 = ijk = -1$. R^m respects the set of all real column vectors with m coordinates. $R^{m\times n}$, $C^{m\times n}$, $Q^{m\times n}$ and $IQ^{m\times n}$ respect the sets of all $m \times n$ real matrices, complex matrices, quaternion matrices and pure imaginary quaternion matrices, respectively. For $A \in C^{m\times n}$, $Re(A)$ and $Im(A)$ denote the real part and the imaginary part of A, respectively. For arbitrary matrix A, A^T, $\bar{A}$, A^H, $tr(A)$ and $A^\dagger$ represent the transpose, the conjugate, the conjugate transpose, the trace and the Moore–Penrose inverse of A, respectively. The identity matrix of order n is denoted by I_n. For any $A, B \in Q^{m\times n}$, we define the inner product $\langle A, B\rangle = tr(B^H A)$. Then $Q^{m\times n}$ is a Hilbert inner product space and the norm of a matrix generated by this inner product is the matrix Frobenius norm $\|\cdot\|$. Let $A = (a_1, a_2, \ldots, a_n) \in R^{m\times n}$, where $a_i \in R^m$ is the ith column of the matrix A, $i = 1, 2, \ldots, n$, and the vec operator of A is defined to be $vec(A) = (a_1^T, a_2^T, \ldots, a_n^T)^T \in R^{mn}$. For $A = (a_{ij}) \in R^{m\times n}$, $B = (b_{ij}) \in R^{p\times q}$, the symbol $A \otimes B = (a_{ij}B) \in R^{mp\times nq}$ stands for the Kronecker product of A and B. The $rand(m)$ is a randomly generated matrix of order m.

In this paper, we will study three kinds of special solutions of the following quaternion matrix equation

$$AXB + CXD = E, \tag{1.1}$$

which were previously discussed in [29] by using the complex representations of quaternion matrices.

Problem 1. Let $A \in Q^{m\times n}, B \in Q^{k\times s}, C \in Q^{m\times n}, D \in Q^{k\times s}, E \in Q^{m\times s}$ and

$$Q_L = \{X | X \in Q^{n\times k},\ \|AXB + CXD - E\| = \min\}. \tag{1.2}$$

Find out $X_Q \in Q_L$ such that $\|X_Q\| = \min_{X\in Q_L} \|X\|$.

Problem 2. Let $A \in Q^{m\times n}, B \in Q^{k\times s}, C \in Q^{m\times n}, D \in Q^{k\times s}, E \in Q^{m\times s}$ and

$$I_L = \{X | X \in IQ^{n\times k},\ \|AXB + CXD - E\| = \min\}. \tag{1.3}$$

Find out $X_I \in I_L$ such that $\|X_I\| = \min_{X\in I_L} \|X\|$.

Problem 3. Let $A \in Q^{m\times n}, B \in Q^{k\times s}, C \in Q^{m\times n}, D \in Q^{k\times s}, E \in Q^{m\times s}$ and

$$R_L = \{X | X \in R^{n\times k},\ \|AXB + CXD - E\| = \min\}. \tag{1.4}$$

Find out $X_R \in R_L$ such that $\|X_R\| = \min_{X\in R_L} \|X\|$.

X_Q, X_I and X_R are respectively called the minimal norm least squares solution, pure imaginary least squares solution, and real least squares solution of the quaternion matrix equation (1.1).

This paper is organized as follows. In Section 2, we state some preliminary results. In Section 3, we study the solutions of Problems 1–3 by using the real representations of quaternion matrices, and compare with those derived in [29] by using the complex representations of quaternion matrices. In Section 4, we provide numerical algorithms for solving Problems 1–3 by using the results obtained in Section 3. In Section 5, we present two numerical examples to illustrate the efficiency of our methods. In Section 6, we offer some concluding remarks.

2. Preliminaries

A quaternion matrix $A \in Q^{m\times n}$ can be uniquely expressed as $A = A_0 + A_1 i + A_2 j + A_3 k$, where $A_0, A_1, A_2, A_3 \in R^{m\times n}$. A pure imaginary quaternion matrix $B \in Q^{m\times n}$ can be uniquely expressed as $B = B_1 i + B_2 k + B_3 j$, where $B_1, B_2, B_3 \in R^{m\times n}$. For any matrix $A = A_0 + A_1 i + A_2 j + A_3 k \in Q^{m\times n}$, its real representation matrix can be defined as

$$A^R = \begin{pmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & A_2 \\ A_2 & A_3 & A_0 & -A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{pmatrix} \in R^{4m\times 4n}.$$