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Group classification for isothermal drift flux model of two phase flows

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ABSTRACT

In this paper, a full symmetry group classification for isothermal multiphase drift flux model is presented. All invariant functions are developed for the Lie algebra, which play a vital role in construction of optimal systems. Further, with the help of one dimensional optimal classification group, invariant solutions are obtained which describe the asymptotic behavior of general solution.

1. Introduction

The study of group invariant solutions of differential equations was first introduced by Sophus Lie in the latter part of nineteenth century. The original aim of Lie was to develop a general theory for the integration of ordinary differential equations (ODEs) similar to the theory given by Galois and Abel for algebraic equations. The fundamental basis of the technique is that when a differential equation is invariant under a Lie group of transformations then a reduction transformation exists, i.e., the order of an nth order ODE can be reduced by r if there exits a r dimensional solvable Lie subalgebra corresponding to the Lie group of transformations admitted by the ODE. The technique is valid for both linear as well as nonlinear differential equations. For scalar as well as system of partial differential equations (PDEs), solutions can be obtained by solving PDEs with fewer independent variables than the given PDEs [1–3].

The classical method of reduction of PDEs by using Lie symmetries [4–7] provides an accomplishable way to construct invariant solutions systematically. One can find some examples of reduction of scalar PDEs as well as system of PDEs in [8,9]. Invariant solutions were first introduced by Lie. The importance of these solutions lie in the fact that they usually describe the asymptotic behavior or display the structure of singularities of general solution. Lie developed two ways of finding invariant solutions such as invariant form method and direct substitution method [8]. Over the years the technique has been modified and in 20th century the method of one dimensional reduction of system of PDEs to ODEs by analyzing the relation between the Lie group parameters become very popular among researchers [10–12]. One of the demerits of this method is that there is no proper logic behind the different cases of reductions, i.e., one has to choose arbitrary relations among the parameters of Lie group for this purpose. The most elegant, logical and useful method of reduction is that classification of optimal subalgebra for the Lie algebra admitted by given PDE. Lie was aware of the concept of adjoint representation of a Lie group on its Lie algebra but its use in classifying group invariant solutions was first introduced by Ovsiannikov [13].

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Ovsiannikov presented the construction of the one-dimensional optimal system for the Lie algebra using a global matrix for the adjoint transformation. Olver [9] slightly modified the construction of one-dimensional optimal system by using adjoint representation instead of global matrix and the classification has done based on Killing form. Recently, Hu et al. [14] developed a technique of finding one-dimensional optimal system with the help of more invariant functions including Killing form. Also authors indicated the way to show the correctness of representative selected for each subclass. Further, the question arises that whether these representatives are mutually inequivalent or not. This question was unanswered until Chou and Qu [15,16] introduced some invariant functions including numerical and conditional invariants to address the inequivalence among the elements in the optimal system.

Apart from one dimensional optimal system, researchers are interested to study two dimensional as well as higher dimensional optimal systems which are useful for higher dimensional problems with more than two independent variables. Two dimensional optimal classification first described by Ovsiannikov [13] and he has chosen the first element, say w_1, of each two dimensional optimal system from one dimensional classification and another element, say w_2, so that w_1 and w_2 form a two dimensional subalgebra. Further, Galas et al. [17] refined this selection by showing that w_2 must be an element of $\mathrm{Nor}(w_1)/w_1$, where $\mathrm{Nor}(w_1)/w_1$ denotes the Normalizer of w_1 coset w_1. One can see the rigorous study of higher dimensional optimal classifications in [15,18,19].

In fluid mechanics, two phase flow is a flow of gas and liquid usually in pipes. The most commonly studied cases of two phase flows are in large scale power systems. Coal and gas fired power stations used very large boilers to produce steam for use in turbines. In such cases, pressurized water is passed through heated pipes and it changes to steam as it moves through the pipe. The design of boilers requires a detailed understanding of two phase flow heat transfer. Even more critically, nuclear reactors use water to remove heat from the reactor core using two phase flows. Due to the presence of surface tension and some other factors two phase flow problems become nonlinear and are quite complex to solve analytically and numerically because of the physics, thermodynamics of each phase, and the large number of waves related to the hyperbolic character. Hence, it is convenient to apply Lie group analysis, to obtain invariant solutions, for such models. In our study we consider isothermal no-slip drift flux model for multiphase flows in a network of pipes [20] and our aim is to obtain similarity solutions by using symmetry analysis.

The sketch of the present work is as follows. In Section 2, Lie symmetries are obtained for the isothermal drift flux model. In Section 3, analysis of Lie symmetries are discussed which includes structure of Lie algebra, construction of invariant functions and one to five dimensional optimal classifications. Further, in Section 4 some invariant solutions are obtained by using Lie symmetry reductions for each class of one dimensional optimal subalgebra which is followed by conclusions in Section 5.

2. Lie symmetries

The isothermal no-slip drift flux model [20] for multiphase flows in a network of pipes can be represented in the form

$$\begin{aligned}
&\rho_{1t} + \rho_1 u_x + u\rho_{1x} = 0,\\
&\rho_{2t} + \rho_2 u_x + u\rho_{2x} = 0,\\
&(\rho_1 + \rho_2)(u_t + uu_x) + a^2(\rho_{1x} + \rho_{2x}) = 0
\end{aligned} \tag{1}$$

where ρ_1, ρ_2 are the density of phase 1 (gas), phase 2 (liquid) respectively, u is the common velocity, x is spatial variable, t is time variable and a is the constant which depends on both phases.

Here our aim is to construct Lie group of point transformations which leaves the governing system of PDEs (1) invariant. Such transformations are useful to reduce system of PDEs to system of ODEs as well as to find invariant solutions.

The one parameter (ϵ) infinitesimal Lie group of transformations, which leaves the system (1) invariant, is given by

$$\begin{aligned}
x^* &= x + \epsilon\xi(x, t, u, \rho_1, \rho_2) + O(\epsilon^2),\\
t^* &= t + \epsilon\eta(x, t, u, \rho_1, \rho_2) + O(\epsilon^2),\\
\rho_1^* &= \rho_1 + \epsilon\tau(x, t, u, \rho_1, \rho_2) + O(\epsilon^2),\\
\rho_2^* &= \rho_2 + \epsilon\phi(x, t, u, \rho_1, \rho_2) + O(\epsilon^2),\\
u^* &= u + \epsilon\psi(x, t, u, \rho_1, \rho_2) + O(\epsilon^2)
\end{aligned} \tag{2}$$

where ξ, η, τ, ϕ and ψ, infinitesimals of the Lie group of transformations (2), are to be determined.

Following the straightforward analysis mentioned in [8] we obtain the infinitesimals as follows

$$\xi = \alpha_1 + \alpha_2 x + \alpha_3 t, \qquad \eta = \alpha_4 + \alpha_2 t, \qquad \tau = \alpha_5\rho_1, \qquad \phi = \alpha_5\rho_2, \qquad \psi = \alpha_3,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_5$ are arbitrary constants.

The infinitesimal generators corresponding to each parameter α_i are given by

$$X_1 = \frac{\partial}{\partial x}(\text{translation in } x)$$