A diffusive toxin producing phytoplankton model with maturation delay and three-dimensional patch

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ABSTRACT

The dynamics of a diffusive toxin producing phytoplankton model with maturation delay and three-dimensional patch subject to Neumann boundary condition is investigated in this paper. The global stability of boundary equilibrium is obtained. The local stability of the coexistent equilibrium and the existence of Hopf bifurcation are investigated. The conditions for determining the bifurcation direction and the stability of the bifurcating periodic solution are derived.

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1. Introduction

Phytoplankton are the basis for aquatic food chains and they can absorb not less than half of the carbon dioxide, producing a huge amount of oxygen for human and other living animals. The importance of phytoplankton in marine ecosystem has been widely recognized [1–4].

In the ocean, phytoplankton may be eaten by zooplankton. In reality, to avoid being eaten by zooplankton, phytoplankton may use various anti-grazing strategies such as filamentous structures [5], cell morphology [6], and toxin releasing [7]. Among these anti-grazing strategies, the toxin releasing is an important anti-grazing strategy, and it has effect on both the phytoplankton and zooplankton species.

In the last decades, many scholars have constructed different mathematical models to describe the dynamics of toxin-producing phytoplankton [8–11]. In [12], Chattopadhyay et al. propose a toxin-producing phytoplankton model with three-dimensional patch for explaining both red tides and recurring phytoplankton blooms, that is

$$
\begin{cases}
\dfrac{dP}{dt} = rP\left(1 - \dfrac{P}{K}\right) - \dfrac{cfPZ}{a + \gamma P}, \\
\dfrac{dZ}{dt} = \dfrac{efPZ}{a + \gamma P} - \mu Z - e\rho P^{2/3}Z,
\end{cases}
\tag{1.1}
$$

with the initial conditions $P(0) = P_0 > 0, Z(0) = Z_0 > 0$, which are biologically meaningful. Here $P(t)$ and $Z(t)$ represent phytoplankton and zooplankton densities at time t respectively. All the parameters in the model are positive. r and K represent intrinsic growth rate and the carrying capacity of the prey respectively. The functional response is Holling type II.

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f represents the rate of free phytoplankton that can be preyed by zooplankton. c and e represent the predation rate and the conversion rate respectively. μ is the natural mortality of zooplankton. ρ is the measure of toxicity. In [12], authors suppose that a fraction $1-f$ of the phytoplankton can form three-dimensional patch in the ocean and the surface of the patch results proportional to $\rho P^{2/3}$. To know the meaning of these parameters, one can refer to [12]. Authors study the stability and Hopf bifurcation for system (1.1) and qualitatively analyze the model by simulation [12].

In recent years, phytoplankton and zooplankton models with diffusion term derive great attention, since that phytoplankton and zooplankton distribute inhomogeneous in different spatial location at time t in the lakes or oceans, and they will move or diffuse for many reasons, such as currents and turbulent diffusion. So in more realistic ecological models, the diffusion should be considered. In this paper, we will study the effect of diffusion on system (1.1). In addition, time delay is an important factor in phytoplankton and zooplankton model and it may affect the dynamics of the model. Considering the time delay in maturation of zooplankton, we consider the following model

$$
\begin{cases}
\dfrac{\partial P(x,t)}{\partial t} = d_1 \Delta P + rP\left(1 - \dfrac{P}{K}\right) - \dfrac{cfPZ}{a + \gamma P}, & x \in (0, \Omega),\ t > 0 \\[2mm]
\dfrac{\partial Z(x,t)}{\partial t} = d_2 \Delta Z + \dfrac{efP(t-\tau)Z}{a + \gamma P(t-\tau)} - \mu Z - e\rho P^{2/3}Z, & x \in (0, \Omega),\ t > 0 \\[2mm]
P_x(0,t) = Z_x(0,t) = 0, \qquad P_x(\Omega, t) = Z_x(\Omega, t) = 0, & t > 0 \\[2mm]
P(x,\theta) = P_0(x,\theta) \geq 0, \qquad Z(x,\theta) = Z_0(x,\theta) \geq 0, & x \in [0, \Omega],\ \theta \in [-\tau, 0].
\end{cases}
\tag{1.2}
$$

In system (1.2), we choose Neumann boundary condition based on the hypothesis that the region phytoplankton and zooplankton lived is closed with no species entering and leaving it at the boundary. In the rest of this paper, we will assume $\Omega = l\pi$, where $l > 0$.

The rest of this paper is organized as follows. In Section 2, we study global stability of the boundary equilibrium. In Section 3, we study the stability of coexistent equilibria and existence of Hopf bifurcation. In Section 4, we investigate the stability and direction of bifurcating periodic solution. In Section 5, we give some numerical simulations. In Section 6, we give a conclusion.

2. Global stability of the boundary equilibrium

Obviously, system (1.2) has a origin equilibrium $(0, 0)$ which is unstable and a boundary equilibrium $(K, 0)$. In this section, we discuss the global stability of the boundary equilibrium $(K, 0)$. Consider the case $f < \mu(a + \gamma K)/(eK)$. Denote

$$
g_1(\phi, \psi) = r\phi_1\left(1 - \frac{\phi_1}{K}\right) - \frac{cf\phi_1\phi_2}{a + \gamma\phi_1}, \qquad g_2(\phi, \psi) = \frac{ef\psi_1\phi_2}{a + \gamma\psi_1} - \mu\phi_2 - e\rho\phi_1^{2/3}\phi_2,
$$

where $\phi = (\phi_1, \phi_2)^T$, $\psi = (\psi_1, \psi_2)^T$. It is easy to see that $g = (g_1, g_2)$ is mixed quasi-monotone in $\bar{\mathbb{R}}_+^2 \times \bar{\mathbb{R}}_+^2$. Define $(\hat{P}, \hat{Z}) = (0, 0)$ and $(\tilde{P}, \tilde{Z}) = (M_1, M_2)$, where $M_1 \geq K$ and $M_2 \geq efM_1/(a + \gamma M_1)$. Then $(\hat{P}, \hat{Z})$ and $(\tilde{P}, \tilde{Z})$ are coupled upper and lower solutions of the system (1.2), since

$$
r\tilde{P}\left(1 - \frac{\tilde{P}}{K}\right) - \frac{cf\tilde{P}\hat{Z}}{a + \gamma\tilde{P}} \leq 0, \qquad \frac{ef\tilde{P}\tilde{Z}}{a + \gamma\tilde{P}} - \mu\tilde{Z} - e\rho\hat{P}^{2/3}\tilde{Z} \leq 0,
$$

$$
r\hat{P}\left(1 - \frac{\hat{P}}{K}\right) - \frac{cf\hat{P}\tilde{Z}}{a + \gamma\hat{P}} \geq 0, \qquad \frac{ef\hat{P}\hat{Z}}{a + \gamma\hat{P}} - \mu\hat{Z} - e\rho\tilde{P}^{2/3}\hat{Z} \geq 0.
$$

Choose M_1 and M_2 as sufficiently large, from Theorem 2.1 in [13], we know that there exists a unique global nonnegative solution (P, Z) for system (1.2) with nonnegative initial value

$$
P_0(x, \theta), \qquad Z_0(x, \theta), \quad x \in [0, L\pi],\ \theta \in [-\tau, 0]
$$

and $P_0(x, \theta) \not\equiv 0$ and $Z_0(x, \theta) \not\equiv 0$. Furthermore, the maximum principle implies that $P(x, t), Z(x, t) > 0$ for $t > 0$.

Let $(\hat{P}, \hat{Z}) = (\epsilon, 0)$ and $(\tilde{P}, \tilde{Z}) = (K, \delta(\epsilon))$, where ϵ is a small positive number and $\delta(\epsilon) = r(1 - \epsilon/K)(a + \gamma\epsilon)/cf$. It is easy to verify that $(\epsilon, 0)$ and $(K, \delta(\epsilon))$ are also coupled upper and lower solutions of the system (1.2). When

$$
\epsilon < \phi_1, \qquad \psi_1 < K, \qquad 0 < \phi_2, \qquad \psi_2 < \delta(\epsilon)
$$

from the boundedness of the partial derivative of g_i ($i = 1, 2$) with respect to ϕ, ψ, we know that g_i satisfy the Lipschitz condition. We denote the Lipschitz constants by K_i, ($i = 1, 2$). From Theorem 2.1 in [13], we see that there exists a unique global solution (P, Z) to the system (1.2) and it satisfies $(\epsilon, 0) \leq (P, Z) \leq (K, \delta(\epsilon))$ whenever $(\epsilon, 0) \leq (P_0(x, \theta), Z_0(x, \theta)) \leq (K, \delta(\epsilon))$.