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Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

# A saddle point least squares approach for primal mixed formulations of second order PDEs

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ARTICLE INFO

*Article history:*
Received 10 May 2016
Received in revised form 15 September 2016
Accepted 10 November 2016
Available online xxxx

*Keywords:*
Least squares
Saddle point systems
Mixed methods
Uzawa type algorithms
Conjugate gradient
Dual DPG

ABSTRACT

We present a Saddle Point Least Squares (SPLS) method for discretizing second order elliptic problems written as primal mixed variational formulations. A stability LBB condition and a data compatibility condition at the continuous level are automatically satisfied. The proposed discretization method follows a general SPLS approach and has the advantage that a discrete inf–sup condition is automatically satisfied for standard choices of the test and trial spaces. For the proposed iterative processes a nodal basis for the trial space is not required. Efficient preconditioning techniques that involve inversion only on the test space can be considered. Stability and approximation properties for two choices of discrete spaces are investigated. Applications of the new approach include discretization of second order problems with highly oscillatory coefficient, interface problems, and higher order approximation of the flux for elliptic problems with smooth coefficients.



## 1. Introduction

The SPLS method for discretizing variational formulations with different types of test and trial spaces was introduced in [1]. The method is related with the Bramble–Pasciak least squares approach introduced in [2]. The SPLS method combines known theory and discretization techniques for approximating elliptic problems with theory and techniques for solving symmetric saddle point problems, [3–14]. It provides a unified framework for discretizing variational formulations of PDEs formulated as first order differential equations or systems. Both the test and trial spaces for the SPLS discretization are conforming finite element spaces. The *test space* is *chosen first*, and the discrete *trial space* is built from the *test space, second*, using the action of the continuous differential operator (associated with the given problem) on the *test space*. For the proposed method, assembly of stiffness matrices for the trial spaces is avoided. A detailed review of the SPLS method is presented in the next section. In [1] the method is designed for discretization of *first order systems* such as div – curl systems. The main goal of the paper is to show that (SPLS) approach can be applied to efficiently solve primal mixed formulations for *second order elliptic problems*.

The paper is organized as follows. In Section 2, we introduce notation and review the SPLS approach as presented in [1]. In addition, we present stability and approximation properties for two choices of discrete spaces. In Section 3, we apply the general method to approximate solutions of second order elliptic problems with variable coefficients. In Section 4, we present numerical results obtained via SPLS discretization. After the conclusion of Section 5, we include Appendix, where the proposed Uzawa type iterative solvers are reviewed.

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http://dx.doi.org/10.1016/j.camwa.2016.11.014
0898-1221/

## 2. The general SPLS approach

The general problem that can be discretized using a SPLS is:
Find $p \in Q$ such that

$$b(v,p) = \langle f, v\rangle, \quad \text{for all } v \in V \quad \text{or} \quad B^*p = f, \tag{2.1}$$

where $V$ and $Q$ are infinite dimensional Hilbert spaces and $b(\cdot,\cdot)$ is a continuous bilinear form on $V \times Q$, that satisfies a standard inf – sup condition and $f \in V^*$. We assume that on $V$ and $Q$ the inner products $a_0(\cdot,\cdot)$ and $(\cdot,\cdot)$ induce the norms $|\cdot|_V = |\cdot| = a_0(\cdot,\cdot)^{1/2}$ and $\|\cdot\|_Q = \|\cdot\| = (\cdot,\cdot)^{1/2}$, respectively. The dual pairings on $V^* \times V$ and $Q^* \times Q$ are denoted by $\langle\cdot,\cdot\rangle$. Here, $V^*$ and $Q^*$ denote the duals of $V$ and $Q$, respectively. With the inner products $a_0(\cdot,\cdot)$ and $(\cdot,\cdot)$, we associate the operators $\mathcal{A} : V \to V^*$ and $\mathcal{C} : Q \to Q^*$ defined by

$$\langle \mathcal{A}u, v\rangle = a_0(u,v) \quad \text{for all } u,\ v \in V$$

and

$$\langle \mathcal{C}p, q\rangle = (p,q) \quad \text{for all } p,\ q \in Q.$$

The operators $\mathcal{A}^{-1} : V^* \to V$ and $\mathcal{C}^{-1} : Q^* \to Q$ are the Riesz-canonical isometries. We assume that $b(\cdot,\cdot)$ satisfies

$$\sup_{p\in Q}\ \sup_{v\in V} \frac{b(v,p)}{\|p\|\,|v|} = M < \infty, \tag{2.2}$$

and that the following inf – sup condition holds,

$$\inf_{p\in Q}\ \sup_{v\in V} \frac{b(v,p)}{\|p\|\,|v|} = m > 0. \tag{2.3}$$

With the form $b$, we associate the linear operators $B : V \to Q^*$ and $B^* : Q \to V^*$ defined by

$$\langle Bv, q\rangle = b(v,q) = \langle B^*q, v\rangle \quad \text{for all } v \in V,\ q \in Q.$$

It is known that under the assumption (2.3), the operator $\mathcal{C}^{-1}B : V \to Q$ is onto, see e.g., [15]. We let $V_0$ be the kernel of $B$ or $\mathcal{C}^{-1}B$, i.e.,

$$V_0 = \mathrm{Ker}(B) = \{v \in V \,|\, Bv = 0\} = \{v \in V \,|\, \mathcal{C}^{-1}Bv = 0\}.$$

The existence and the uniqueness of (2.1) were first studied by Aziz and Babuška in [3]. It is well known that if a bounded form $b : V \times Q \to \mathbb{R}$ satisfies (2.3) and the data $f \in V^*$ satisfies the *compatibility condition*

$$\langle f, v\rangle = 0, \quad \text{for all } v \in V_0, \tag{2.4}$$

then, the problem (2.1) has a unique solution, see e.g. [3,15].

With the problem (2.1) we associate the SPLS formulation:
Find $(u,p) \in (V,Q)$ such that

$$\begin{array}{lllll} a_0(u,v) & + & b(v,p) & = \langle f, v\rangle & \text{for all } v \in V, \\ b(u,q) & & & = 0 & \text{for all } q \in Q. \end{array} \tag{2.5}$$

The following statement summarizes the connection between the two variational formulations.

**Proposition 2.1.** *In the presence of the continuous* inf – sup *condition* (2.3) *and the compatibility condition* (2.4)*, we have that* $p$ *is the unique solution of* (2.1) *if and only if* $(u = 0, p)$ *is the unique solution of* (2.5).

### 2.1. SPLS discretization

Due to Proposition 2.1, the *SPLS discretization* of (2.1) is defined as a standard saddle point discretization of (2.5). We let $V_h \subset V$ and $M_h \subset Q$ be finite dimensional approximation spaces and consider the restrictions of the forms $a_0(\cdot,\cdot)$ and $b(\cdot,\cdot)$ to the discrete spaces $V_h$ and $M_h$. Assume that the following discrete inf – sup condition holds for the pair $(V_h, M_h)$.

$$\inf_{p_h\in M_h}\ \sup_{v_h\in V_h} \frac{b(v_h,p_h)}{\|p_h\|\,|v_h|} = m_h > 0. \tag{2.6}$$

We define $V_{h,0}$ to be the kernel of the discrete operator $B_h$, i.e.,

$$V_{h,0} := \{v_h \in V_h \,|\, b(v_h,q_h) = 0,\ \text{for all } q_h \in M_h\}$$

and let $V_{h,0}^{\perp}$ denote the orthogonal complement of $V_{h,0}$ with respect to $a_0(\cdot,\cdot)$ inner product on $V_h$.

Please cite this article in press as: C. Bacuta, K. Qirko, A saddle point least squares approach for primal mixed formulations of second order PDEs, Computers and Mathematics with Applications (2016), http://dx.doi.org/10.1016/j.camwa.2016.11.014